Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras — April 14, 2019
Instructor: Abhishek Sinha

Problem Set 6

• This problem set is due on **April 30, 2019**.
• Each problem carries 10 points.
• You may work on the problems in groups of size at most **two**. However, **each student must write their own solution**. If you collaborate on the problems, clearly mention the name of your collaborator.

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1. **(Moments vs. Chernoff Bounds)** Show that moment bounds for tail probabilities are always better than Chernoff bounds. More precisely, let X be a nonnegative random variable and let $t > 0$. The best moment bound for the tail probability $\mathbb{P}(X \geq t)$ is $\min_q \mathbb{E}(X^q)t^{-q}$ where the minimum is taken over all positive integers. The best Chernoff bound is $\inf_{s>0} \mathbb{E}\big(\exp(s(X-t))\big)$. Prove that

$$\min_q \mathbb{E}(X^q)t^{-q} \leq \inf_{s>0} \mathbb{E}\big(\exp(s(X-t))\big).$$

2. **(Discrete Loomis-Whitney Inequality)** Let A denote a finite subset of $\mathbb{Z}^d$ and let A_i denote the $d-1$ dimensional projection of A along the i^{th} coordinate. Show that

$$|A|^{d-1} \leq \prod_{i=1}^{d} |A_i|.$$

HINT: Use Han's inequality with uniform distribution on A.

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3. **(Maximum Eigenvalue of a Random Matrix)** In this problem we will use Talagrand's inequality to derive *exponential* concentration of the maximum eigenvalue of a random symmetric matrix around its median value.
For $1 \leq i \leq j \leq n$ let x_{ij} be independent real variables with $|x_{ij}| \leq 1$, and set $x_{ji} = x_{ij}$. Let X be a real $n \times n$ symmetric matrix with entries (x_{ij}). Let $\lambda_1(X)$ be the maximum Eigenvalue of X and let M be the median of $\lambda_1(X)$. We will show that

$$\mathbb{P}(|\lambda_1(X) - M| \geq t) \leq 4\exp(-t^2/32).$$

   (a) Define the sets $A = \{X : \lambda(X) \leq M\}$, $B = \{Y : \lambda(Y) \geq M + t\}$. For any $Y \in B$ let v denote its (unit-norm) eigenvector corresponding to the top eigenvalue. Show that for all $X \in A$:

$$v^T(Y - X)v \geq t.$$

(b) Show that there exists an a with $||a|| \leq 1$ such that:

$$\frac{1}{2\sqrt{2}} v^T(Y - X)v \leq d_a(x, y).$$

(c) Combine part (a) and (b) and finish the proof by applying Talagrand's inequality to the sets A and B.

4. **(Disjoint Triangles in Random Graphs)** Consider a random graph G on n vertices in which edges appear independently each with probability $p(n) = \frac{\lambda}{n}$. Let $\Delta(G)$ denote the maximum number of edge-disjoint triangles in G. Using Talagrand's concentration inequality show that:

$$\mathbb{P}(\Delta(G) \leq m - t\sqrt{3m}) \leq 2e^{-t^2/4},$$

where m is the median of $\Delta(G)$.

5. **(Concentration of Convex Lipschitz functions)** Let $\{X_i\}_{i=1}^n$ be independent random variables, each supported on the interval $[a, b]$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a coordinate-wise convex and L-Lipschitz differentiable function with respect to the Euclidean norm.

(a) Show that

$$\sum_i (f(\boldsymbol{x}) - f(\boldsymbol{x}^{(i)}))^2 \leq L^2(b - a)^2,$$

where $f(\boldsymbol{x}^{(i)}) = \inf_{x_i'} f(x_1, x_2, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$.

(b) Using an appropriate Logarithmic Sobolev inequality and Herbst argument, show that for all $t > 0$:

$$\mathbb{P}(f(\boldsymbol{X}) \geq \mathbb{E}(f(\boldsymbol{X})) + t) \leq \exp\left(- \frac{t^2}{4L^2(b - a)^2} \right).$$

Note: A naive bound using only the Lipschitz condition yields

$$\sum_{i=1}^{n} \left(f(\boldsymbol{x}) - f(\boldsymbol{x}^{(i)}) \right)^2 \leq nL^2(b - a)^2.$$

The convexity assumption results in a huge improvement over this simple bound.
