Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras March 25, 2020
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Problem Set 4

- This problem set is due on **April 10, 2020**. Please email your solutions to the TA.
- Each problem carries 10 points.
- You must work on the problems by yourself. No collaboration allowed.

1. **(Symmetric RVs)** Let $\{X_i\}_{i=1}^n$ be n independent Rademacher random variables, *i.e.*, $\mathbb{P}\big[X_i = \pm 1\big] = \frac{1}{2}$. Consider the random variable $Z_n = \sum_{i=1}^n r_i X_i$, where $\{r_i\}_{i=1}^n$ are arbitrary real constants. Prove that Z_n is symmetric, and that

$$\mathbb{E}\big[Z_n^4\big] \leq 3(\mathbb{E}\big[Z_n^2\big])^2.$$

2. **(Large deviation for sum of products)** Suppose X_i is an i.i.d. zero mean sequence of random variables with a finite everywhere moment generating function $M(\theta) = \mathbb{E}(\exp(\theta X_1))$. Argue the existence and express the following large deviations limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Bigg(\sum_{1 \leq i \neq j \leq n} X_i X_j > n^2 z \Bigg)$$

in terms of $M(\theta)$ and z.

3. **(Number of Power-Constrained Ternary Codewords)** Consider a ternary codeword of length n, consisting of n symbols, each taken from the set $\{0, 1, 2\}$. This codeword is to be transmitted over a power-constrained wireless link. Assume that the symbols $0, 1, 2$ takes $0, 1, 2$ unit of energy respectively, for transmission. Due to the power-constraint, only those sequences are eligible for transmission whose required average transmission energy is at most $\frac{1}{2}$.

   In other words, for a sequence $(s_1, s_2, \ldots, s_n)$, if the i^{th} transmitted symbol s_i consumes w_i amount of energy, then the codeword is eligible for transmission if and only if

$$\frac{1}{n} \sum_{i=1}^{n} w_i \leq \frac{1}{2}.$$

   Let $N(n)$ denote the number of eligible codewords of length n. Define

$$l = \lim_{n\to\infty} \frac{1}{n} \ln(N(n)).$$

   Show that the above limit exists and compute l.

4. **(Link with time-varying capacity)** Consider calls accessing a link with time-varying capacity. The call holding times are independent and exponentially distributed with mean $\frac{1}{\mu}$, and each call requires one unit of bandwidth from the link to maintain QoS. The link has an available capacity of nc_0 in the time interval $[0, \tau_1)$, an available capacity of nc_1 in $[\tau_1, \tau_2)$, and an available capacity of nc_2 in $[\tau_2, \infty]$, where $\tau_2 > \tau_1 > 0$. We assume that $nc_0 > nc_1 > nc_2$ and $n > 0$. Suppose that there are $\lfloor n\alpha \rfloor$ calls in progress at time $t = 0$ (where $\alpha \in (0, c_0)$), and that no further calls ever arrive at the system. Let $N(t)$ denote the number of calls in progress at time t. Thus, the bandwidth requested by the calls at time t is $N(t)$. We are interested in estimating the probability that the required bandwidth ever exceeds the available capacity, up to a logarithmic equivalence. To this end, prove the following fact for appropriate functions $I_i(x)$:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\exists t \in [0, \infty) \text{ such that } N(t) \geq c(t)) = -\min_{1,2} I_i(c_i/\alpha),$$

where $c(t)$ is the available capacity at time t. Explicitly compute the functions $I_i(x)$.