Problem Set 4

- This problem set will be due on **November 7, 2021**.
- Each problem carries 10 points.
- Collaboration is **strictly prohibited**. Each student must submit their own work.

1. **(Existence of an Invariant Distribution)** The Markov-Kakutani Theorem asserts that, for any convex compact subset C of $\mathbb{R}^n$ and any linear continuous mapping T of C into C, T has a fixed point (in the sense that $T(x) = x$ for some $x \in C$). Use this to prove that a finite stochastic matrix $\boldsymbol{P}$ has a non-zero left eigenvector corresponding to the eigenvalue 1, i.e., the equation

$$\pi \boldsymbol{P} = \pi$$

has a non-zero solution.

2. **(A Symmetric Markov Chain)** Let X be a Markov Chain with state space $S = \{0, 1, 2\}$ and transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1-p & p & 0 \\ 0 & 1-p & p \\ p & 0 & 1-p \end{pmatrix},$$

where $0 < p < 1$. Prove that

$$\boldsymbol{P}^n = \begin{pmatrix} a_{0n} & a_{1n} & a_{2n} \\ a_{2n} & a_{0n} & a_{1n} \\ a_{1n} & a_{2n} & a_{0n} \end{pmatrix}.$$

where $a_{0n} + \omega a_{1n} + \omega^2 a_{2n} = (1 - p + p\omega)^n$, ω being a complex cube root of unity.

3. **(Transient Chains)** Let $\{X_n, n \geq 0\}$ be an irreducible DTMC on $\mathcal{S} = \{0, 1, 2, \ldots\}$. For $j \in \mathcal{S}$, let M_j denote the number of visits to state j for $n \geq 1$.

   (a) Write down an expression for $\mathbb{P}(M_j = k | X_0 = i)$ in terms of f_{ij} and f_{jj}.

   (b) Obtain $\mathbb{E}(M_j | X_0 = i)$ in terms of f_{ij} and f_{jj}.

   (c) Hence show that if $\{X_n\}$ is transient, then for all $i, j \in \mathcal{S}$, $\lim_{k \to \infty} p_{ij}^{(k)} = 0$.

4. **(Queue-length stability in wireless channels)** Consider a simple model of a discrete-time wireless channel: the channel is ON with probability μ and OFF with probability $1 - \mu$; when the channel is ON, it can serve one packet in the slot, and, when the channel is OFF, the channel cannot serve any packets. Packets arrive at this wireless channel according to the following arrival process: a maximum of one arrival occurs in each instant. The probability of an arrival in the current slot is 0.8 if there was an arrival in the previous time slot. The probability of an arrival in the current slot is 0.1 if there was no arrival in the previous time slot. Unserved packets wait in a queue till they get served.

   (a) For what values of μ would you expect this system to be stable?

   HINT: The arrival process is a simple two-state Markov Chain. Find the mean arrival rate of the packet arrival process, assuming that it is in steady state. The mean service rate must be larger than this mean arrival rate.

   (b) For these values of μ and assuming that the arrival process is stationary, show that an appropriate Markov Chain describing the state of the queueing system is stable (i.e., positive recurrent) using the Foster-Lyapunov Theorem.

   HINT: The state of the Markov Chain describing the queueing system is the number of packets in the queue along with the state of the arrival process.

5. **(Symmetric Random Walks)** Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. Define the process $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i.$$

   (a) Is $\{S_n\}_{n\geq 1}$ a Markov Chain? Identify its transition probabilities.

   (b) Show that $p_{00}^{(2n)} \sim \Theta(\frac{1}{\sqrt{n}})$. What can you conclude about the transience/recurrence of the chain?

   Use Stirling's approximation $n! = \sqrt{2\pi n}(\frac{n}{e})^n(1 + O(\frac{1}{n}))$.

   (c) Let S_n^1, S_n^2, and S_n^3 be three independent copies of S_n. Show that the chain (S_n^1, S_n^2) is recurrent and the chain (S_n^1, S_n^2, S_n^3) is transient.