Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras March 14, 2019
Instructor: Abhishek Sinha

Problem Set 4

- This problem set is due on **March 28, 2019** in the class.
- Each problem carries 10 points.
- You may work on the problems in groups of size at most **two**. However, **each student must write their own solution**. If you collaborate on the problems, clearly mention the name of your collaborator.

1. **(Convergence of Likelihood Ratios)** Consider a sequence of independent tosses of a coin and let $\mathbb{P}(H)$ be the probability of a head in any toss. Let $\mathcal{A}$ be the hypothesis that $\mathbb{P}(H) = \alpha$ and let $\mathcal{B}$ be the hypothesis that $\mathbb{P}(H) = \beta, 0 < \alpha, \beta < 1$. Let X_i denote the outcome of the i^{th} toss and let

   $$Z_n = \frac{\mathbb{P}(X_1, X_2, \ldots, X_n|\mathcal{A})}{\mathbb{P}(X_1, X_2, \ldots, X_n|\mathcal{B})}.$$

   Show that if $\mathcal{B}$ is true, then:

   (a) $\{Z_n\}_{n\geq 1}$ is a Martingale w.r.t. the natural filtration.

   (b) $\lim_{n\to\infty} Z_n$ exists with probability 1.

   (c) If $b \neq a$, what is $\lim_{n\to\infty} Z_n$?

2. **(Conditionally Poisson Random Variables)** Let $\{Z_n, n \geq 1,\}$ be a sequence of random variables such that $Z_1 \equiv 1$ and given $Z_1, Z_2, \ldots, Z_{n-1}$, Z_n is a Poisson random variable with mean $Z_{n-1}, \forall n \geq 1$. What can we say about Z_n for large enough n?

3. **(Markov's version of Maximal Inequality for Supermartingales)** In the class, we derived Doob-Kolmogorov's maximal inequality for *sub*martingales. In this problem, you will derive a similar concentration inequality for *super*martingales.

   (a) Using Fatou's lemma, show that if X is a non-negative supermartingale and T is a stopping time (not necessarily finite *a.s.*), then

   $$\mathbb{E}(X_T \mathbb{1}(T < \infty)) \leq \mathbb{E}(X_0).$$

   (b) Hence, deduce the following stronger form of Markov's inequality for supermartingles

   $$\mathbb{P}(\sup_n X_n \geq c) \leq \frac{\mathbb{E}(X_0)}{c}, \quad \forall c > 0. \tag{1}$$

4. **(Cantelli's version of Maximal Inequality for Martingales)** Let $(\boldsymbol{Y}, \boldsymbol{\mathcal{F}})$ be a martingale with $\mathbb{E}(Y_0) = 0$, and $\mathbb{E}(Y_n^2) < \infty, \forall n$. Show that

$$\mathbb{P}(\max_{1 \le k \le n} Y_k > x) \le \frac{\mathbb{E}(Y_n^2)}{\mathbb{E}(Y_n^2) + x^2}, \quad x > 0.$$

HINT: Use Doob-Kolomogorov's inequality.
