Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras February 25, 2020
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Problem Set 3

- This problem set is due on **March 11, 2020** in the class.
- Each problem carries 10 points.
- No collaboration allowed. Each student must write his/her own solution.

1. **(Stopped σ-fields)** Let $\boldsymbol{\mathcal{F}}$ be a filtration. For any stopping time T with respect to $\boldsymbol{\mathcal{F}}$, denote by $\mathcal{F}_T$ the collection of all events A such that, for all n, $A \cap \{T \leq n\} \in \mathcal{F}_n$. Let S and T be stopping times with respect to $\boldsymbol{\mathcal{F}}$.
   (a) Show that $\mathcal{F}_T$ is a σ-field, and that T is measurable with respect to $\mathcal{F}_T$.

   (b) If $A \in \mathcal{F}_S$, show that $A \cap \{S \leq T\} \in \mathcal{F}_T$.

   (c) Let S and T satisfy $S \leq T$. Show that $\mathcal{F}_S \subseteq \mathcal{F}_T$

2. **("How to gamble if you must")** Suppose that you are gambling in one of the famous casinos at Goa (although, I won't recommend it!). Your winnings per unit stake on game n are ϵ_n, where the ϵ_n are i.i.d. random variables with

$$\mathbb{P}(\epsilon_n = +1) = p, \mathbb{P}(\epsilon_n = -1) = q, \quad \frac{1}{2} < p = 1 - q < 1.$$

   You can't borrow money from others during the game. Thus, your stake W_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n-1$. Your objective is to maximize the expected "interest rate" $\mathbb{E}\log(Z_N/Z_0)$, where N is a given integer representing the length of the game, and Z_0, your fortune at time 0, is a given constant. Let $\{W_n\}_n$ be any previsible strategy and $D_{\text{KL}}(\boldsymbol{p}_1||\boldsymbol{p}_2)$ denote the Kullback-Leibler (KL) Divergence [1] between the distributions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$.

   (a) Show that, under $\{W_n\}_{n\geq 1}$, $\{\log Z_n - n\alpha\}_{n\geq 1}$ is a *super*martingale, where

$$\alpha = D_{\text{KL}}\big((p,q)||(1/2,1/2)\big) \equiv p\log p + q \log q + \log 2,$$

   so that $\mathbb{E}\log(Z_N/Z_0) \leq N\alpha$.

   (b) Hence, find the optimal strategy.

3. **(Expected Polynomial-time solvability of 2-SAT)** In this problem, we will analyze a simple algorithm for satisfiability.

   (a) Consider a supermartingale $\{X_t\}_{t\geq 1}$ that takes values in $\{0, 1, 2, \ldots, n\}$, with $X_0 = s$. Set $D_t := X_t - X_{t-1}, t \geq 1$ and assume that for all $t \geq 1$ we have $\mathbb{E}(D_{t+1}^2|\mathcal{F}_t) \geq \sigma^2$,

[1] For discrete distributions $D_{\text{KL}}(\boldsymbol{p}||\boldsymbol{q}) = \sum_i p_i \log \frac{p_i}{q_i}$.

for some constant σ^2. We are interested in T, the number of steps needed for X_t to reach 0. Show that
$$\mathbb{E}(T) \leq \frac{2ns - s^2}{\sigma^2} \leq \frac{n^2}{\sigma^2}.$$
(b) Recall the classic Boolean Satisfiability problem [2]. Show that the following simple randomized polynomial time algorithm will find a satisfying assignment (given that it exists) in expected quadratic time.
Given a 2-CNF formula ϕ with n variables, pick an arbitrary initial assignment a_0. If ϕ is not satisfied by a_0, pick an arbitrary unsatisfied clause C_0. Choose a literal of C_0 uniformly at random and flip the value of that variable to obtain assignment a_1. Using the result of part (a), show that the algorithm will find a satisfying assignment (given that it exists) after at most $O(n^2)$ rounds in expectation.

4. **(Polya's Urn)** A bag contains red and blue balls, with initially r red and b blue ones where $rb > 0$. A ball is drawn from the bag, its colour noted, and then it is returned to the bag together with a new ball of the same colour. Let R_n be the number of red balls after n such operations.
(a) Show that $Y_n = \frac{R_n}{n+r+b}$ is a Martingale which converges almost surely and in mean.
(b) Let T be the number of balls drawn until the first blue ball appears, and suppose that $r = b = 1$. Show that $\mathbb{E}[(T+2)^{-1}] = \frac{1}{4}$.
(c) Suppose $r = b = 1$, and show that $\mathbb{P}(Y_n \geq \frac{3}{4} \text{ for some } n) \leq \frac{2}{3}$.

5. **(Size of the maximum matching in a random bipartite graph)** Given $1 \leq d \leq n$, let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be disjoint sets of cardinality n, and let G be a bipartite random graph with vertex set $U \cup V$, such that if V_i denotes the set of neighbors of u_i, then $V_1, V_2, \ldots, V_n$ are independent, and each is uniformly distributed over the set of all $\binom{n}{d}$ subsets of V of cardinality d. A matching for G is a subset of edges M such that no two edges in M have a common vertex. Let Z denote the maximum of cardinalities of the matchings in G.

   (a) Show that $\mathbb{E}(Z) = \Theta(n)$.
   (b) Give an upper bound on $\mathbb{P}(|Z - \mathbb{E}[Z]| \geq \gamma\sqrt{n})$, for $\gamma > 0$, showing that for fixed d, the distribution of Z is concentrated about its mean as $n \to \infty$.

[2] `https://www2.cs.duke.edu/courses/fall14/compsci330/notes/scribe23.pdf`