EE5110: Probability Foundations for Electrical Engineers
Indian Institute of Technology Madras September 22, 2021
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Problem Set 3

- This problem set is due on **October 6, 2021** in the class.
- Each problem carries 10 points.
- Collaboration is **strictly prohibited**. Each student must submit their own work.

1. **(Integrals)** Rigorously evaluate the following integrals (taken in the sense of Lebesgue):
   (a) $\lim_{n\to\infty} \int_0^\infty (1+\frac{x}{n})^{-n} \sin(x/n) dx$.
   (b) $\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)} dx$.

2. **(On the growth of the maximum of n independent exponentials)** Suppose that $X_1, X_2, \ldots$ are independent random variables, each with the exponential distribution with parameter $\lambda = 1$. For $n \geq 2$, let $Z_n = \frac{\max(X_1, X_2, \ldots, X_n)}{\ln(n)}$.
   (a) Find a simple expression for the CDF of Z_n.
   (b) Show that the sequence $\{Z_n\}_{n\geq 1}$ converges in distribution to a constant, and find the constant.

3. **(Dominated Convergence)** Suppose $|X_n| \leq Z, \forall n \geq 1$, where $\mathbb{E}(Z) < \infty$. Prove that if $X_n \xrightarrow{p.} X$ then $\lim_{n\to\infty} \mathbb{E}|X_n - X| = 0$.

4. **(Convergence in total variation)** The sequence of discrete random variables X_n taking non-negative integral values with mass functions p_n, is said to *converge in total variation* to X with mass function p if
$$\sum_x |p_n(x) - p(x)| \to 0, \text{ as } n \to \infty.$$
   Suppose $X_n \to X$ in total variation, and $u : \mathbb{R} \to \mathbb{R}$ is bounded. Show that $\mathbb{E}(u(X_n)) \to \mathbb{E}(u(X))$.

5. **(Convergence of a random recursion)** Suppose $\{U_n\}_{n\geq 1}$ are independent random variables, each independently distributed on the interval $[0, 1]$. Let $X_0 = 0$, and define the sequence of random variables $\{X_n, n \geq 1\}$ by the following recursion:
$$X_n = \max\left\{X_{n-1}, \frac{X_{n-1} + U_n}{2}\right\}.$$
   - (**Warm Up:**) Simulate the random sequence $\{X_n\}_{n\geq 1}$ (in any high-level programming language of your choice) for $1 \leq n \leq 1000$ for a sufficiently many times. What do you observe?

- Does $\lim_{n\to\infty} X_n$ exist in the a.s. sense? Rigorously prove your result.
- Does $\lim_{n\to\infty} X_n$ exist in the m.s. sense? Rigorously prove your result.
- Identify the random variable Z such that $X_n \to Z$ in probability. Justify your answer.