Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras February 24, 2019
Instructor: Abhishek Sinha

Problem Set 3

- This problem set is due on **March 14, 2019** in the class.
- Each problem carries 10 points.
- You may work on the problems in groups of size at most **two**. However, **each student must write their own solution**. If you collaborate on the problems, clearly mention the name of your collaborator.

1. **(Martingales)** On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider a sequence of random variables $X_1, X_2, \ldots, X_n$ and σ-fields $\mathcal{F}_1, \ldots, \mathcal{F}_n \subset \mathcal{F}$ such that $\mathbb{E}[X_j|\mathcal{F}_{j-1}] = X_{j-1}$ and $\mathbb{E}[X_j^2] < \infty$.

   (a) Prove directly (without using Jensen's inequality) that $\mathbb{E}[X_j^2] \geq \mathbb{E}[X_{j-1}^2]$ for all $j = 2, \ldots, n$.

   (b) Suppose $X_n = X_1$ almost surely. Prove that in this case $X_1 = \ldots = X_n$ almost surely.

2. **("How to gamble if you must")** Suppose that you are gambling in one of the famous casinos at Goa (although, I shall strictly advise you against it!). Your winnings per unit stake on game n are ϵ_n, where the ϵ_n are i.i.d. random variables with

$$\mathbb{P}(\epsilon_n = +1) = p, \mathbb{P}(\epsilon_n = -1) = q, \quad \frac{1}{2} < p = 1 - q < 1.$$

   You can't borrow money from others during the game. Thus, your stake W_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n-1$. Your objective is to maximize the expected "interest rate" $\mathbb{E}\log(Z_N/Z_0)$, where N is a given integer representing the length of the game, and Z_0, your fortune at time 0, is a given constant. Let $\{W_n\}_n$ be any previsible strategy and $D_{\text{KL}}(\boldsymbol{p}_1||\boldsymbol{p}_2)$ denote the Kullback-Leibler (KL) Divergence [1] between the distributions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$.

   (a) Show that, under $\{W_n\}_{n\geq 1}$, $\{\log Z_n - n\alpha\}_{n\geq 1}$ is a *super*martingale, where

$$\alpha = D_{\text{KL}}\big((p,q)||(1/2,1/2)\big) \equiv p\log p + q\log q + \log 2,$$

   so that $\mathbb{E}\log(Z_N/Z_0) \leq N\alpha$.

   Hint: Recall Gibb's inequality, which states that for any two distributions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, we have $D_{\text{KL}}(\boldsymbol{p}_1||\boldsymbol{p}_2) \geq 0$, with the equality holding iff $\boldsymbol{p}_1 = \boldsymbol{p}_2$.

[1] For discrete distributions $D_{\text{KL}}(\boldsymbol{p}||\boldsymbol{q}) = \sum_i p_i \log \frac{p_i}{q_i}$.

(b) Hence, find the optimal strategy.

(c) [*Purely optional. Attempt only if you are familiar with the theory of MDP. However, if you can correctly solve this part, you'll get a bonus of* 5 *points.*] Solve Bellman's equations to find the optimal strategy using an MDP formulation of the problem. Compare this strategy to that of part (b).

HINT: Think about how to rewrite the total reward as a sum of per-stage rewards.

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3. **(Size of the maximum matching in a random bipartite graph)** Given $1 \leq d \leq n$, let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be disjoint sets of cardinality n, and let G be a bipartite random graph with vertex set $U \cup V$, such that if V_i denotes the set of neighbors of u_i, then $V_1, V_2, \ldots, V_n$ are independent, and each is uniformly distributed over the set of all $\binom{n}{d}$ subsets of V of cardinality d. A matching for G is a subset of edges M such that no two edges in M have a common vertex. Let Z denote the maximum of cardinalities of the matchings in G.

   (a) Show that $\mathbb{E}(Z) = \Theta(n)$.

HINT: Consider the case $d = 1$ and use the linearity of expectation. When can a vertex $v \in V$ be included in a matching?

(b) Give an upper bound on $\mathbb{P}(|Z - \mathbb{E}[Z]| \geq \gamma\sqrt{n})$, for $\gamma > 0$, showing that for fixed d, the distribution of Z is concentrated about its mean as $n \to \infty$.

HINT: Use Mcdiarmid's inequality with the i.i.d. random variables $V_i, 1 \leq i \leq n$.
