Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras March 1, 2021
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Problem Set 2

- This problem set is due on **March 15, 2021**.
- Each problem carries 10 points.
- Collaboration is **not permitted**. Each student must submit his/her own work.

1. **(Random Walk)** Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with

$$\mathbb{P}(X_1 = 1) = p, \mathbb{P}(X_1 = -1) = 1 - p,$$

where $0 \leq p \leq 1$ and $p \neq \frac{1}{2}$. Let $S_0 = 0$ and $S_n = S_{n-1} + X_n, n \geq 1$. For each $n \geq 1$, define the event $A_n = \{\omega : S_n(\omega) = 0\}$ and let $A = \limsup A_n$. Let $\mathcal{F}_\infty$ be the tail σ-algebra corresponding to the sequence of r.v.s $\{X_n\}_{n\geq 1}$. Show that

(a) $A \notin \mathcal{F}_\infty$.

(b) Nonetheless, $\mathbb{P}(A) \in \{0, 1\}$.

HINT: Use Stirling's approximation.

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2. **(Conditional Expectation)**

(a) Let the random variables $\{Z_n\}_{n\geq 1}$ be independent, each with finite mean. Let $X_0 = a$, and $X_n = a + Z_1 + Z_2 + \ldots + Z_n$ for $n \geq 1$. Prove that

$$\mathbb{E}\big(X_{n+1} | \sigma(X_1, X_2, \ldots, X_n)\big) = X_n + \mathbb{E}(Z_{n+1}).$$

(b) Suppose that $X, Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}(X|\sigma(Y)) = Y, \mathbb{E}(Y|\sigma(X)) = X \quad \text{a.s.}$$

Show that $X = Y$ almost surely.

(c) **(Linear Estimation)** Let $X_1, X_2, \ldots, X_n$ be random variables with zero expectations and covariance matrix $\boldsymbol{V}$ [1]. Using the orthogonality principle, find the linear map $h(\cdot)$ of $\{X_i\}_{i=1}^n$ which minimizes the mean squared error $\mathbb{E}\{(Y - h(X_1, X_2, \ldots, X_n))^2\}$.

[1] This means that $V_{ij} = \mathbb{E}(X_i X_j), 1 \leq i, j \leq n$.

3. **(Kolmogorov-Hajek-Renyi inequality)**

   (a) Show that if X is a non-negative supermartingale and T is a stopping time, then

   $$\mathbb{E}(X_T; T < \infty) \leq \mathbb{E}(X_0).$$

   Hence or otherwise, show that $\mathbb{P}(\sup_n X_n \geq c) \leq \mathbb{E}(X_0)/c$.

   (b) Let $\{Z_n\}_{n \geq 0}$ be a Martingale sequence with $Z_0 = 0$ and let $\{v_j\}_{j \geq 0}$ be a sequence of non-decreasing constants with $v_0 = 0$. Prove that

   $$\mathbb{P}(|Z_j| \leq v_j, \quad \forall 1 \leq j \leq n) \geq 1 - \sum_{j=1}^{n} \mathbb{E}(Z_j - Z_{j-1})^2 / v_j^2.$$

   Hint: Define the stopping time N to be the first time for which $|Z_N| > v_N$. Let it be equal to n if $|Z_j| \leq v_j, \forall 1 \leq j \leq n$. Now analyze the corresponding stopped Martingale.

4. **(Randomized polynomial-time solvability of 2-SAT)** In this problem, we will analyze a simple randomized algorithm for $2-$satisfiability.

   (a) Consider a supermartingale $\{X_t\}_{t \geq 1}$ that takes values in $\{0, 1, 2, \ldots, n\}$, with $X_0 = s$. Set $D_t := X_t - X_{t-1}, t \geq 1$ and assume that for all $t \geq 1$ we have $\mathbb{E}(D_{t+1}^2 | \mathcal{F}_t) \geq \sigma^2$, for some constant σ^2. We are interested in T, the number of steps needed for X_t to reach 0. Show that

   $$\mathbb{E}(T) \leq \frac{2ns - s^2}{\sigma^2} \leq \frac{n^2}{\sigma^2}.$$

   (b) Recall the classic Boolean Satisfiability problem [2]. Show that the following simple randomized polynomial time algorithm will find a satisfying assignment (given that it exists) in expected quadratic time.
   Given a 2-CNF formula ϕ with n variables, pick an arbitrary initial assignment a_0. If ϕ is not satisfied by a_0, pick an arbitrary unsatisfied clause C_0. Choose a literal of C_0 uniformly at random and flip the value of that variable to obtain assignment a_1. Using the result of part (a), show that the algorithm will find a satisfying assignment (given that it exists) after at most $O(n^2)$ rounds in expectation.

5. **(Controlling a spaceship)**

   (a) Imagine that you are the captain of a spaceship currently located at a distance of R_0 from the solar system. Your objective is to steer the spaceship into the solar system, which is assumed to be a ball of radius $r < R_0$ centered around the Sun located at the origin. You can set the distance to be traveled by the space-ship in each hop based on the available information. However, due to some unfortunate mechanical failures, you can no longer specify the direction of movement. As

[2] `https://www2.cs.duke.edu/courses/fall14/compsci330/notes/scribe23.pdf`

a result, at every hop, the spaceship moves in a direction chosen uniformly at random over a 3D sphere of a radius of your choice from its previous location. Let R_n be the distance from the Sun to your spaceship after n hops. Show that irrespective of the control strategy adopted, the sequence $\{\frac{1}{R_n}\}_{n\geq 1}$ is a super-martingale and that for any strategy which always sets a distance no greater than that from Sun to your spaceship, $\{\frac{1}{R_n}\}_{n\geq 1}$ constitutes a Martingale sequence. Hence or otherwise, show that

$$\mathbb{P}[\text{The spaceship enters into the solar system}] \leq r/R_0.$$

Hint: You may need to use a classic result from high-school physics on the gravitational potential of a uniform spherical shell to evaluate an integral appearing in this problem.

(b) Assume that after working on the faulty engine for hours, you have been able to confine the random motion of the spaceship along a fixed plane passing through the Sun. However, the next hop-length is now automatically set to the current distance to the Sun. In other words, if the distance to the Sun from the spaceship after the n^{th} hop is R_n, in the next hop, the spaceship is moved uniformly at random on a circle with center at the current location and radius R_n. Show that the spaceship gets into the solar system almost surely.

Hint: Define the r.v.s $V_n = \ln R_n - \ln R_{n-1}, n \geq 1$. Show that $\{V_n\}_{n\geq 1}$ are i.i.d. with zero mean and non-zero but finite variance. Now consider the summation $S_n = \sum_{k=1}^{n} V_k$. Using CLT, show that $\mathbb{P}(\inf_n S_n < \ln(r_0/R_0)) = 1$ and conclude the result.