Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras — February 8, 2019
Instructor: Abhishek Sinha

Problem Set 2

- This problem set is due on **February 21, 2019** in the class.
- Each problem carries 10 points.
- You may work on the problems in groups of size at most **two**. However, **each student must write their own solution**. If you collaborate on the problems, clearly mention the name of your collaborator.

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1. **(Consecutive Heads)** This is an extension of a problem we discussed in the class in connection with the Borel-Cantelli Lemmas. Consider a sequence of independent, fair coin tossing, and let H_n be the event that the n^{th} coin toss is head. Determine the following probabilities:

   (a) $\mathbb{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lceil 2\log_2 n \rceil} \text{ i.o.})$

   (b) $\mathbb{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lceil \log_2 n \rceil} \text{ i.o.})$

2. **(Random Walk)** Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with

$$\mathbb{P}(X_1 = 1) = p, \mathbb{P}(X_1 = -1) = 1 - p,$$

   where $0 \leq p \leq 1$ and $p \neq \frac{1}{2}$. Let $S_0 = 0$ and $S_n = S_{n-1} + X_n, n \geq 1$. For each $n \geq 1$, define the event $A_n = \{\omega : S_n(\omega) = 0\}$ and let $A = \limsup A_n$. Let $\mathcal{F}_\infty$ be the tail σ-algebra corresponding to the sequence of r.v.s $\{X_n\}_{n\geq 1}$. Show that

   (a) $A \notin \mathcal{F}_\infty$.

   (b) Nonetheless, $\mathbb{P}(A) \in \{0, 1\}$.

   HINT: Use Stirling's approximation for the factorial of a number.

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3. **($\mathcal{L}^2$ vs. $\mathcal{L}^1$)**

   (a) Let X_1, X_2 be i.i.d. non-negative random variables such that $X_1 \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ but $X_1 \notin \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Let $Y = \min\{X_1, X_2\}$. Show that $Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

   (b) (**Generalization**) Let $X_1, X_2, \ldots, X_n$ be a collection of n i.i.d. non-negative random variables such that $X_1 \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ but $X_1 \notin \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Let Y be the *Second largest* random variable in the above collection [1]. Show that $Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

[1] In order-statistic notation $Y = X_{(n-1)}$.

4. **(Conditional Expectation)**

   (a) Let the random variables $\{Z_n\}_{n\geq 1}$ be independent, each with finite mean. Let $X_0 = a$, and $X_n = a + Z_1 + Z_2 + \ldots + Z_n$ for $n \geq 1$. Prove that

   $$\mathbb{E}\big(X_{n+1}|\sigma(X_1, X_2, \ldots, X_n)\big) = X_n + \mathbb{E}(Z_{n+1}).$$

   (b) Suppose that $X, Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ such that

   $$\mathbb{E}(X|\sigma(Y)) = Y, \mathbb{E}(Y|\sigma(X)) = X \quad \text{a.s.}$$

   Show that $X = Y$ almost surely.

   (c) **(Linear Estimation)** Let $X_1, X_2, \ldots, X_n$ be random variables with zero expectations and covariance matrix $\boldsymbol{V}$ [2]. Using the orthogonality principle, find the linear map $h(\cdot)$ of $\{X_i\}_{i=1}^n$ which minimizes the mean squared error $\mathbb{E}\{(Y - h(X_1, X_2, \ldots, X_n))^2\}$.

[2] This means that $V_{ij} = \mathbb{E}(X_i X_j), 1 \leq i, j \leq n$.