Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras January 23, 2020
Instructor: Abhishek Sinha

Problem Set 1

- This problem set is due on **February 6, 2020** in the class.
- Each problem carries 10 points.
- Collaboration is **not permitted**. Each student must submit their own work.

1. **(Minimum Expectation)** Let the random variable $X \in [0, 1]$ have a continuous density function $p(x)$. In other words, for any Borel set $B \in \mathbb{B}([0, 1])$, we have

$$\mathbb{P}(X \in B) = \int_B p(x)dx,$$

where the integral is taken in the usual Riemann sense. Assume that the density $p(x)$ is *non-decreasing* in $[0, 1]$. Note that $p(\cdot)$ needs not be differentiable. Find the minimum value of $\mathbb{E}_p(X)$ over all such feasible density functions $p(\cdot)$.

2. **(Density of Sets)** A set $A \subset \mathbb{N}$ is said to have **asymptotic density** θ if

$$\lim_{n\to\infty} |A \cap \{1, 2, \ldots, n\}|/n = \theta.$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? If not, is it an algebra?

3. **(Limsup of Events)** Suppose that the sequence of events $\{A_n\}_{n\geq 1}$ satisfy

$$\mathbb{P}(A_n) \to 0 \text{ and } \sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty.$$

Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$.

4. **(Strong Law for Normal r.v.s)** (a) Prove that if G is a random variable with the normal $N(0, 1)$ distribution, then, for $x > 0$,

$$\mathbb{P}(G > x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2/2}dx \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}.$$

(b) Let $X_1, X_2, \ldots$ be a sequence of independent $N(0, 1)$ variables. Prove that, with probability 1, $L \leq 1$, where

$$L := \limsup(X_n/\sqrt{2\log n}).$$

(c) Let $S_n := X_1 + X_2 + \ldots + X_n$. Recall that $S_n/\sqrt{n}$ has the $N(0, 1)$ distribution. Prove that

$$\mathbb{P}(|S_n| < \sqrt{2n\log n}, \text{ev}) = 1.$$

5. **('Cauchy Sequence')** Let's call a sequence of random variables $\{X_n\}_{n\geq 1}$ to be *Cauchy in probability* if for all $\epsilon > 0$, there exists an N_ϵ such that $\mathbb{P}(|X_m - X_n| > \epsilon) < \epsilon$ for $m, n > N_\epsilon$.
(a) Using the first Borel-Cantelli Lemma show that there is a subsequence $\{X_{n_k}\}_{k\geq 1}$ and a random variable X such that $\lim_k X_{n_k} = X$ w.p. 1.
(b) Conclude that $X_n \to_p X$.

6. **(Infinite Sum)** Let $\delta, \epsilon > 0$, and let $X_1, X_2, \ldots$ be a sequence of non-negative random variables such that $\mathbb{P}(X_i \geq \delta) \geq \epsilon$ for all $i \geq 1$. Prove that with probability one, $\sum_{i=1}^{\infty} X_i = \infty$.