Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras February 10, 2021
Instructor: Abhishek Sinha

Problem Set 1

• This problem set is due on **February 24, 2021**. Please email your solution (preferably typeset in LaTeX) to Samrat.
• Each problem carries 10 points.
• Collaboration is **not permitted**.

1. (**Symmetrization**) For a set T, let $\big(X_1(t), X_2(t), \ldots X_N(t)\big)$ be a collection of N zero-mean independent random variables indexed by each $t \in T$. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_N$ be independent symmetric Bernoulli random variables. Prove that

$$\frac{1}{2}\mathbb{E}\sup_{t\in T}\sum_{i=1}^{N}\epsilon_i X_i(t) \leq \mathbb{E}\sup_{t\in T}\sum_{i=1}^{N}X_i(t) \leq 2\mathbb{E}\sup_{t\in T}\sum_{i=1}^{N}\epsilon_i X_i(t).$$

HINT: Consider another independent copy of the r.v.s $\boldsymbol{X}(t)$.

2. **(Minimum Expectation)** Let the random variable $X \in [0,1]$ have a continuous density function $p(x)$. In other words, for any Borel set $B \in \mathbb{B}([0,1])$, we have

$$\mathbb{P}(X \in B) = \int_B p(x)dx,$$

Assume that the density $p(x)$ is *non-decreasing* in $[0,1]$. Note that $p(\cdot)$ needs not be differentiable. Find the minimum value of $\mathbb{E}_p(X)$ over all such density functions $p(\cdot)$.

3. **(A Generalization of the first Borel-Cantelli Lemma)** Show that the following statements are equivalent (*i.e.*, $a \implies b$ and $b \implies a$)

(a) For each $\epsilon > 0$, $\exists$ an event A such that $\mathbb{P}(A) \geq 1-\epsilon$ and $\sum_{n=1}^{\infty}\mathbb{P}(A_n \cap A) < \infty$.
(b) $\mathbb{P}(\limsup A_n) = 0$.

4. **(Strong Law for Normal r.v.s)** (a) Prove that if G is a random variable with the normal $N(0,1)$ distribution, then, for $x > 0$,

$$\mathbb{P}(G > x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-x^2/2}dx \leq \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}.$$

(b) Let $X_1, X_2, \ldots$ be a sequence of independent $N(0,1)$ variables. Prove that, with probability 1, $L \leq 1$, where

$$L := \limsup(X_n/\sqrt{2\log n}).$$

(c) Let $S_n := X_1 + X_2 + \ldots + X_n$. Recall that $S_n/\sqrt{n}$ has the $N(0,1)$ distribution. Prove that

$$\mathbb{P}(|S_n| < 2\sqrt{n\log n}, \text{ev}) = 1.$$

5. **('Cauchy Sequence')** Let's call a sequence of random variables $\{X_n\}_{n\geq 1}$ to be *Cauchy in probability* if for all $\epsilon > 0$, there exists an N_ϵ such that $\mathbb{P}(|X_m - X_n| > \epsilon) < \epsilon$ for $m, n > N_\epsilon$.
   (a) Using the first Borel-Cantelli Lemma show that there is a subsequence $\{X_{n_k}\}_{k\geq 1}$ and a random variable X such that $\lim_k X_{n_k} = X$ w.p. 1.
   (b) Conclude that $X_n \to_p X$.

6. **($\mathcal{L}^2$ vs. $\mathcal{L}^1$)** Let $X_1, X_2, \ldots, X_n$ be a collection of n i.i.d. non-negative r.v.s such that $\mathbb{E}(X_1) < \infty$ but $\mathbb{E}(X_1^2) = \infty$. Sort the r.v.s in decreasing order $X_{(1)}(\omega) \geq X_{(2)}(\omega) \geq \ldots \geq X_{(n)}(\omega) \geq 0$. Let the r.v. $Y(\omega) := X_{(2)}(\omega)$ be the second largest in the above collection. Show that $\mathbb{E}(Y^2) < \infty$.