EE5110: Probability Foundations for Electrical Engineers
Indian Institute of Technology Madras — August 13, 2021
Instructor: Abhishek Sinha

Problem Set 1

- This problem set is due on **August 27, 2021** in the class.
- Each problem carries 10 points.
- Collaboration is **strictly prohibited**. Each student must submit their own work.

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1. **(Cardinality of a set)** Recall that, in the class we proved that the set of all binary sequences is uncountable. Hence or otherwise, prove that $2^{\mathbb{N}}$, the power set of the natural numbers, is uncountable.

2. **(Minimum Number of Trials)** Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \ldots$ a point x_{n+1} is chosen uniformly from the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ.

   Hint: Let $N(x)$ be the minimum number of trials necessary when $x_k = x$ for some k. Set up an equation in terms of $N(x)$ and solve it upon supplying the appropriate boundary conditions. Clearly mention if you make any assumptions.

3. **(Limsup of Events)** Suppose that the sequence of events $\{A_n\}_{n\geq 1}$ satisfy

   $$\mathbb{P}(A_n) \to 0 \text{ and } \sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty.$$

   Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$.

4. **(Strong Law for Normal r.v.s)** (a) Prove that if G is a random variable with the normal $N(0,1)$ distribution, then, for $x > 0$,

   $$\mathbb{P}(G > x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2/2} dx \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}.$$

   (b) Let $X_1, X_2, \ldots$ be a sequence of independent $N(0,1)$ variables. Prove that, with probability 1, $L \leq 1$, where

   $$L := \limsup (X_n/\sqrt{2\log n}).$$

   (c) Let $S_n := X_1 + X_2 + \ldots + X_n$. Recall that $S_n/\sqrt{n}$ has the $N(0,1)$ distribution. Prove that

   $$\mathbb{P}(|S_n| < 2\sqrt{n \log n}, \text{ev}) = 1.$$

5. **('Cauchy Sequence')** Let's call a sequence of random variables $\{X_n\}_{n\geq 1}$ to be *Cauchy in probability* if for all $\epsilon > 0$, there exists an N_ϵ such that $\mathbb{P}(|X_m - X_n| > \epsilon) < \epsilon$ for $m, n > N_\epsilon$.
(a) Using the first Borel-Cantelli Lemma show that there is a subsequence $\{X_{n_k}\}_{k\geq 1}$ and a random variable X such that $\lim_k X_{n_k} = X$ w.p. 1.
(b) Conclude that $X_n \to_p X$.