Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology, Madras January 22, 2019
Instructor: Abhishek Sinha

Problem Set 1

- This problem set is due on **February 5, 2019** in the class.
- Each problem carries 10 points.
- You may work on the problems in groups of size at most **two**. However, **each student must write their own solution**. If you collaborate on the problems, clearly mention the name of your collaborator.

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1. **(Limsup of the Events)** Suppose the events A_n satisfy $\mathbb{P}(A_n) \to 0$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$.

2. **(Strong Law for Normal R.V.s)** (a) Prove that if G is a random variable with the normal $N(0, 1)$ distribution, then, for $x > 0$,
$$\mathbb{P}(G > x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2/2} dx \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}.$$
(b) Let $X_1, X_2, \ldots$ be a sequence of independent $N(0, 1)$ variables. Prove that, with probability 1, $L \leq 1$, where
$$L := \limsup(X_n/\sqrt{2 \log n}).$$
(c) Let $S_n := X_1 + X_2 + \ldots + X_n$. Recall that $S_n/\sqrt{n}$ has the $N(0, 1)$ distribution. Prove that
$$\mathbb{P}(|S_n| < \sqrt{2n \log n}, \text{ev}) = 1.$$

3. **("Violation" of the Strong Law)** Let $X_1, X_2, \ldots$ be independent RV such that
$$X_n = \begin{cases} n^2 - 1, & \text{with probability } n^{-2}. \\ -1, & \text{otherwise.} \end{cases}$$
Prove that
   (a) $\mathbb{E}(X_n) = 0, \forall n$.
   (b) If $S_n = \sum_i^n X_i$, then
$$\frac{S_n}{n} \to -1, \text{ a.s.}$$

4. **(Infinite Sum)** Let $\delta, \epsilon > 0$, and let $X_1, X_2, \ldots$ be a sequence of non-negative random variables such that $\mathbb{P}(X_i \geq \delta) \geq \epsilon$ for all $i \geq 1$. Prove that with probability one, $\sum_{i=1}^{\infty} X_i = \infty$.