Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras — March 18, 2021
Instructor: Abhishek Sinha

Mid Term

- This mid-term is due on **April 5, 2021**.
- Each problem carries 10 points.
- No collaboration is allowed for mid-term!

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1. **(Infinite Integrals)** In the following, let $f : [0,1] \to \mathbb{R}$ be continuous. **Rigorously** evaluate the following limits.

$$(1) \qquad \lim_{n\to\infty} \int_0^1 \int_0^1 \ldots \int_0^1 f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) dx_1 dx_2 \ldots dx_n$$

$$(2) \qquad \lim_{n\to\infty} \int_0^1 \int_0^1 \ldots \int_0^1 f\left((x_1 x_2 \ldots x_n)^{1/n}\right) dx_1 dx_2 \ldots dx_n.$$

$$(3) \qquad \lim_{n\to\infty} \int_0^1 \int_0^1 \ldots \int_0^1 \frac{x_1^2 + x_2^2 + \ldots + x_n^2}{x_1 + x_2 + \ldots + x_n} dx_1 dx_2 \ldots dx_n$$

2. **(Superharmonic functions)** A function $\phi : \mathbb{Z}^k \to \mathbb{R}$ on the integer lattice $\mathbb{Z}^k$ is called *superharmonic* if for each lattice point $\boldsymbol{x}$, we have

$$\phi(\boldsymbol{x}) \geq \frac{1}{2k} \sum_{\boldsymbol{y}: ||\boldsymbol{y}-\boldsymbol{x}||_1 = 1} \phi(\boldsymbol{y}),$$

where the sum extends over the $2k$ nearest neighbors. Using Martingale methods, show for $k = 1$ and $k = 2$ that all bounded superharmonic functions are constant.

Hint: You may use the fact that the symmetric random walk on $\mathbb{Z}^k$ is recurrent for $k = 1, 2$.

3. **(Find the odd one out)** Assume that you have m coins. All the coins are unbiased (that is, they have an equal probability of heads or tails), EXCEPT the *special* coin which comes up heads with probability $\frac{1}{2} + \rho$ for some unknown $0 < \rho < \frac{1}{2}$. Your goal is to identify the special coin. For this, you toss each of the coins n times and record the results of each toss. Come up with a decision rule to infer the special coin from the experimental results. Find the sample complexity (*i.e.,* a lower bound to n) of your decision rule to ensure that the probability of error is at most δ.

4. **(Gaussian Complexity of ℓ_0-"balls")** Sparsity plays an important role in many high-dimensional statistical models. The Gaussian complexity of an Euclidean set $S \subseteq \mathbb{R}^d$ is defined as:

$$\mathcal{G}(S) = \mathbb{E}(\max_{\boldsymbol{x}\in S} \sum_{i=1}^d x_i Z_i),$$

where $\{Z_i, 1 \leq i \leq d\}$ are i.i.d. standard Gaussian random variables. In this problem, we will compute the Gaussian complexity of an s-sparse ℓ_0-ball intersected with a unit ℓ_2-ball. Consider the set

$$T^d(s) = \{\theta \in \mathbb{R}^d | ||\theta||_0 \leq s, ||\theta||_2 \leq 1.\}$$

corresponding to all s-sparse vectors contained within the Euclidean unit ball. In this problem, we prove that its Gaussian complexity is upper bounded as:

$$\mathcal{G}(T^d(s)) \precsim \sqrt{s \log\left(\frac{ed}{s}\right)}. \tag{1}$$

(a) First show that $\mathcal{G}(T^d(s)) = \mathbb{E}\big[\max_{|S|=s} ||w_S||_2\big]$, where $w_S \in \mathbb{R}^{|S|}$ denotes the sub-vector of $(w_1, w_2, \ldots, w_d)$ indexed by the subset $S \subseteq \{1, 2, \ldots, d\}$.

(b) Next show that for any fixed subset S of cardinality s:

$$\mathbb{P}\big[||w_S||_2 \geq \sqrt{s} + \delta\big] \leq e^{-\delta^2/2}.$$

(c) Use the preceding parts to establish the bound (1).