EE5110: Probability Foundations for Electrical Engineers
Indian Institute of Technology Madras — September 29, 2021
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Mid-Term

- Each problem carries 10 points.
- To get any credit, **rigorously justify** all of your claims.
- Collaboration is **strictly prohibited**.

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1. **(Independence of Normal Random Variables)** Let X and Z be independent, with $X \sim N(0,1)$, and with $\mathbb{P}(Z=1) = \mathbb{P}(Z=-1) = 1/2$. Define $Y := XZ$ (*i.e.,* Y is the product of X and Z).

   (a) Prove that $Y \sim N(0,1)$.

   (b) Prove that X and Y are *not* independent.

   (c) Prove that $\text{Cov}(X,Y) = 0$.

   (d) It is sometimes claimed that if X and Y are normally distributed random variables with $\text{Cov}(X,Y) = 0$, then X and Y must be independent. Is this claim correct? If not, what should be the correct statement?

2. **(A Stochastic Recursion)** Let D_1 and X be independent and square-integrable random variables such that $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2 > 0$. Define the random variable D_2 as
$$D_2 = \max(0, D_1 + X).$$

   (a) Show that if $-\infty < \mu < 0$ and D_1 and D_2 are identically distributed then
$$d = \mathbb{E}(D_1) = \mathbb{E}(D_2) \leq \frac{\sigma^2}{2|\mu|}.$$

   (b) Show that if on the contrary $\mathbb{E}(X) = \mu > 0$ and $\mathbb{E}D_1 \geq 0$, then $\mathbb{E}(D_2) \geq \mathbb{E}(D_1) + \mu$.

   Hint: You may use the fact that $\big(\max(a,b)\big)^2 \leq a^2 + b^2, \ \forall a, b.$

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3. **(Convergence of Random Variables)** Let $(X_n, n \geq 1)$ be a sequence of i.i.d. random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(X_n = 2) = \mathbb{P}(X_n = 0) = 1/2$ for every $n \geq 1$. Let also $(Y_n, n \geq 1)$ be the sequence of random variables defined as
$$Y_n = \sum_{j=1}^{n} \frac{X_j}{3^j}, \quad n \geq 1.$$

(a) Show that there is a random variable Y such that $Y_n \to Y$ almost surely.

(b) Is it true that $Y_n \overset{\text{m.s.}}{\to} Y$? Justify your answer.

(c) Run a numerical simulation and plot the empirical distribution of Y_n for large enough n.