Mid-Term Test (Take-home)

• This test is due on **March 18, 2020**.
• Each problem carries 10 points.
• You are free to use your notes and textbooks. However, you are not allowed to discuss the problems with anybody else during the test.

1 Consecutive Heads

Consider a sequence of independent, fair coin tossing, and let H_n denote the event that the n^{th} coin toss results in Heads. Determine the following probabilities:
(a) $\mathbb{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lceil 2\log_2 n \rceil} \text{ i.o. })$.
(b) $\mathbb{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lceil \log_2 n \rceil} \text{ i.o. })$.

2 Superharmonic functions

A function $\phi : \mathbb{Z}^k \to \mathbb{R}$ on the integer lattice $\mathbb{Z}^k$ is called *superharmonic* if for each lattice point $\boldsymbol{x}$, we have

$$\phi(\boldsymbol{x}) \geq \frac{1}{2k} \sum_{\boldsymbol{y}:||\boldsymbol{y}-\boldsymbol{x}||_1=1} \phi(\boldsymbol{y}),$$

where the sum extends over the $2k$ nearest neighbors. Using Martingale methods, show for $k = 1$ and $k = 2$ that all bounded superharmonic functions are constant.

3 Bounds for suprema of empirical processes

Consider the random variable $Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^n f(X_i)$ where $\{X_i\}_{i=1}^n$ is an i.i.d. sequence of random variables, and $\mathcal{F}$ is a class of functions taking values in the interval $[0, 1]$. Using the logarithmic Sobolev inequality studied in the class, show that

$$\log \mathbb{E}(e^{\lambda Z}) \leq (e^\lambda - 1)\mathbb{E}(Z), \quad \forall \lambda \geq 0.$$