Topics in Random Processes and Concentrations: EE 6112
Indian Institute of Technology Madras — February 28, 2019
Instructor: Abhishek Sinha

Mid-Term Test (Take-home)

• This test is due on **March 7, 2019** in the class.
• Each problem carries 10 points.
• You are free to use your notes and textbooks. However, you are not allowed to discuss the problems with anybody else during the test.

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1. **(Conditional vs. unconditional expectation)** Let $\mathcal{G}$ be a sub-σ-algebra, and let X and Y be two independent random variables. Give a simple example (by explicitly defining the objects X, Y, and $\mathcal{G}$ of your choice) where

$$\mathbb{E}(XY|\mathcal{G}) \neq \mathbb{E}(X|\mathcal{G})\mathbb{E}(Y|\mathcal{G}).$$

2. **(Independent and exhaustive events)** Let $\{A_n\}_{n\geq 1}$ be a sequence of independent events with $\mathbb{P}(A_n) < 1 \; \forall n$, and $\mathbb{P}(\cup_n A_n) = 1$. Show that $\mathbb{P}(A_n \text{ i.o. }) = 1$.

3. **(Time until elimination)** Players X, Y, and Z play the following game. Initially, each of them has some positive number of coins such that they have a total of s coins among themselves. At each stage, two of them are randomly chosen in sequence, with the first one chosen being required to give 1 coin to the other. All of the possible choices are equally likely, and successive choices are independent of the past. This continues until one of the players has no remaining coins. Show that the expected number of stages until which the game lasts is at most $\frac{s^2}{9}$.

Hint: Define a suitable Martingale and an associated Stopping Time, and use the Optional Stopping Theorem.
