# Optimal Impromptu Deployment of A Multi-Hop Wireless Network On a Random Lattice Path

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by

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# Abstract

We consider the problem of impromptu deployment of a multihop wireless network as a deployment operative steps along a random lattice path, with a Markov evolution. At each step, with various probabilities, the path can either continue in the same direction or can take a turn "North" or "East," or can come to an end, at which point a sensor needs to be placed to send some measurements to a base station near the origin of the path. A decision has to be made at each step whether or not to place a wireless relay node. The problem is motivated by the growing need for emergency first-responders (such as fire-fighters) to deploy sensors for situational awareness (e.g., temperature sensors on fire-doors). Assuming that the measurement traffic generated by the sensor is very low, and simple link-by-link scheduling, we consider the problem of relay placement so as to minimize an end-to-end cost metric, a linear combination of the sum of convex hop costs, and the number of relays placed. In one version, radio propagation is possible only along the straight line segments in the path. In another version, propagation is allowed between any pair of points on the path. In each case, the impromptu relay placement problem is formulated as a total cost Markov decision process. We provide results on the structure of the optimal sequential placement policies, and show how the structure changes with variation of some parameters of the model. We compare the performance of the optimal policy with that of the heuristic of placing a relay when the distance from the previously placed relay exceeds a certain threshold.

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# Chapter 1

# Introduction

The operation of an emergency response team (in situations such as a fire or a terrorist siege) could be facilitated by a wireless sensor network that is deployed as the team moves through the region of operation. Such a network could provide situational awareness, since the team members will not themselves be able to observe all parts of the region of operation. If the layout of the region of operation is not known, then such a network cannot be *planned*, but must be deployed in an "as-you-go" or *impromptu* fashion. In this thesis we are concerned with the rigorous formulation and solution of a simple version of the problem of optimal sequential deployment of a multihop wireless relay network along a random lattice path, as shown in Fig. 1.2. The random lattice path could model a corridor in a large building, or even a trail in a dense forest. networks as *impromptu* wireless networks.

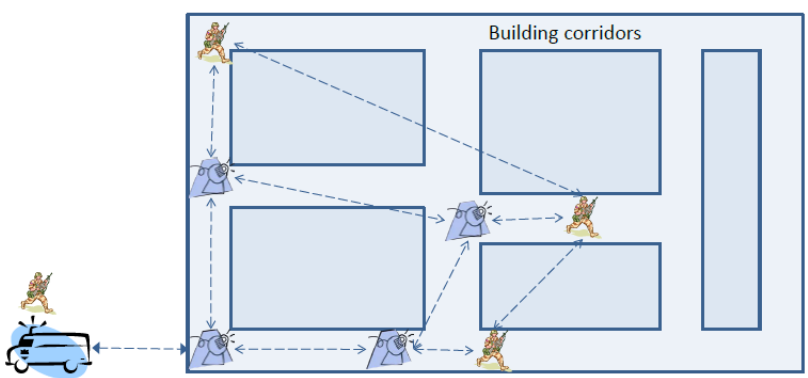



Figure 1.1: An emergency response team being supported by an impromptu wireless sensor network.

### 1.0.1 Related Literature:

While the concept of an impromptu wireless network for first-responders has been around at least since 2001, the literature comprises mainly system architectures, ad hoc algorithms, and deployment experiences.

Aache et al. [1] provide an overview of the on-going European Project called Wireless DEployable Network

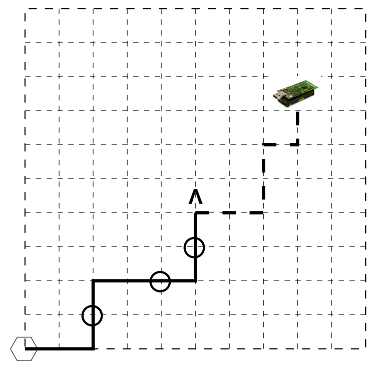



Figure 1.2: A wireless network being deployed as a person steps along a random lattice path. Inverted **V**: location of the deployment person; solid line: path already covered; circles: deployed relays; thick dashed path: the remaining path. The sensor to be placed at the end is also shown

System (WIDENS) which aims at defining a rapidly deployable communication system for public safety or emergency services. The WIDENS project provides an open platform for the validation of adhoc technologies for public safety applications. Portmann et al. [17] describe wireless mesh networks technology and different applications of wireless mesh networks in public safety and disaster situations. This thesis also lists performance requirements of public safety communication systems in the following areas: availability, reliability, survivability, restorability, quality of service and support for prioritization of traffic. To counter the limitation of communicating via a base-station in emergency scenarios, the paper by Gao et al. [9] propose an architecture for an emergency response system relying on a self-configuring wireless mesh network for public safety. The proposed design however requires a control centre which should be installed in the vicinity of the monitoring region and must maintain the connection with all the nodes in the monitoring region. Naudts et al. [15] describe the concept and implementation of a monitoring and planning tool that helps an emergency team in deploying the network and also in providing a real time overview of the status of the network. The authors use the packet pair probing technique for the estimation of the link capacity. Aurisch et al. [3] describe a RSS based relay deployment approach. The emergency responder determines the RSS by radiotap header analysis of the last $N$ beacons. This is compared with a threshold value to determine the quality of a link. These measured values are sent to a command centre which generates a picture of the current communication system status.

To the best of our knowledge, Mondal et al. [13] took the first steps towards rigorously formulating and addressing the problem of optimal deployment of impromptu wireless sensor networks in a simple setting. They called the problem *optimal sequential relay placement (OSRP)*, and formulated and solved OSRP for the problem on a line (a model for a building "corridor") of unknown length. In the problem formulated by Mondal et al. [13], as the deployment operative walks along the corridor, at each step the decision to place or not to place a wireless relay has to be made. The corridor is of unknown length but prior information is available about its probability distribution; at each step, the corridor can come to an end with probability $p$, at which point a sensor has to be placed. Once placed, the sensor sends periodic measurement packets to a base station (BS) (e.g., a situation monitoring truck), which is outside the building, $x$ steps from the entry to the corridor. It is assumed that the measurement rate at the sensor is low, so that (with a very high probability) a packet is delivered to the BS before the next packet is generated at the sensor. We call this the "lone packet model", which is realistic for situations in which the sensor must send a packet every few seconds [20]. The objective of the sequential decision problem is to minimise a certain expected per packet cost (e.g., end-to-end delay or total energy expended by a node), which can be expressed as the sum of the costs over each hop. In practice, the number of relays that can be deployed will be constrained by the ability of the deployment personnel to carry the relays in an emergency situation. Two types of constraints are considered in [13]: expected number of relays constraint, and absolute number of relays constraint. In each case, the problem is formulated as a total cost Markov decision process, and, under the assumption that the per-hop cost function is nonnegative, increasing, convex, and has unbounded derivative, the structures of the optimal policies are characterized.

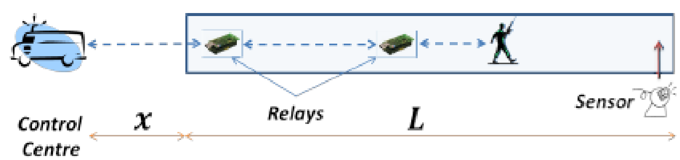



Figure 1.3: Problem studied in [13]. Impromptu placement of wireless relays along a corridor at the end of which a sensor needs to be placed.

*Our Contributions:* In this thesis, while continuing to assume (a) that a single operative moves step-by-step along a line, deciding to place or to not place a relay, (b) that the length of the line is a geometrically distributed random multiple of the step size, (c) that one sensor is placed at the end of the line, (d) that the lone packet traffic model applies, and (d) that the total cost of a deployment is a linear combination of the sum of convex hop costs and the number of nodes placed, we extend the work presented in [13] in two directions:

(i) At each step, the line can take a right angle turn either to the "East" or to the "North" with known fixed probabilites. With the building context in mind, we will sometimes call the path a "corridor," and each straight-line segment (between turns) a "gallery."

(ii) The region through which the line traverses is either radio opaque (i.e., only line-of-site (LOS) radio propagation takes place along the straight line segments), or non-line-of-sight (NLOS) propagation is possible, hence a radio link exists between two nodes placed anywhere on the path; see Fig. 1.4.

For these situations, in this thesis, we provide the following main results:

1. For the LOS case, in Theorem 3.2.2, we establish that the problem can be viewed as a sequence of problems each of which is identical to the one solved in [13]. Further, in Theorem 3.2.6, we show that the one-step-look-ahead (OSLA) rule applies. In Section 3.3 we use the OSLA rule to provide a method for computing the placement

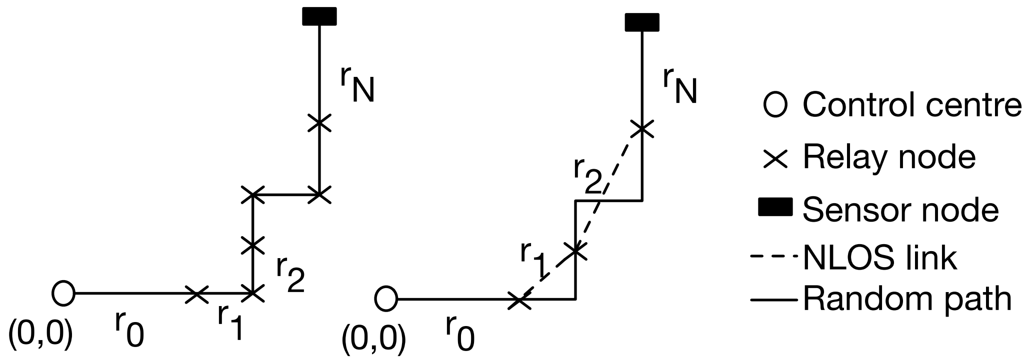



Figure 1.4: A depiction of relay deployment along a random lattice path with LOS (left) and NLOS (right) propagation.

threshold. We have developed an efficient fixed-point-iteration algorithm in section 3.4.2 utilizing the OSLA rule to compute the threshold in an iterative way. We also provide results on the correctness and finite termination properties of our algorithm in that section.

2. For the NLOS case, in Theorem 4.2.2, we show that optimal policy is of the form of a boundary (with respect to the previously placed relay), on crossing which a relay must be placed. In Theorem 4.3.4, we establish that the placement boundary can be characterized via the OSLA rule. Here also, in section 4.6.1, we propose an efficient fixed-point-iteration-algorithm for computing the optimal policy in an iterative way. We also prove its correctness and finite termination properties.

3. In Section 5 we provide several numerical results that illustrate the theoretical development. The relay placement approach proposed in [15, 18, 19, 3], when applied to isotropic propagation as assumed in our thesis, would suggest a distance threshold based placement rule. We numerically obtain the optimal rule in this class, and find that the cost of this policy is numerically indistinguishable from that of the overall optimal policy provided by our theoretical development, thus suggesting that it might suffice to utilize a distance threshold policy. The distance threshold, however, depends on the hop cost function and the parameters of the problem.

# Chapter 2

# System Model and Problem Formulation

## 2.1 System Model

Let $\mathbb{Z}_+$ denote the set of nonnegative integers, and $\mathbb{Z}_+^2$ the nonnegative orthant of the two dimensional integer lattice. We will refer to the x direction as East and the y direction as North. The control centre is located at the origin $(0,0)$. Starting from $(0,0)$ there is a random lattice path that progresses initially Eastward, and takes random turns to the North or to the East (this is to avoid the path to fold back onto itself, see Fig 1.4). Under this restriction, the path evolves as a stochastic process over $\mathbb{Z}_+^2$; a characterization of the stochastic evolution is known to the deployment person (and will be described later in this section).

We assume that the stride length of the deployment person is 1 unit which is the edge length in the lattice. Thus the deployment person starts from $(0,0)$ and moves along the vertices of the path, placing relay nodes at some vertices and finally at the end of the path places a sensor (e.g., a video camera or a temperature sensor). Once placed, the sensor periodically generates measurement packets which are forwarded by the successive relays in a multihop fashion to the control centre.

For two successive relays separated by a distance $r$, we assign a cost of $d(r)$ which could be the average delay incurred over that hop (including transmission overheads and retransmission delays), or the power required to get a packet across the hop. For instance, in our numerical work we use the power cost, $d(r) = P_m + \gamma r^{\eta}$, where $P_m$ is the minimum power required, $\gamma$ represents an SNR constraint and $\eta$ is the path loss exponent. Now suppose $N$ relays are placed such that the successive inter-relay distances are $r_0, r_1, \cdots, r_N$ ($r_0$ is the distance from the control centre at $(0,0)$ and the first relay, and $r_N$ is the distance from the last relay to the sensor placed at the end of the path) then the total cost of this placement is the sum of the one-hop costs $C = \sum_{i=0}^{N} d(r_i)$ (the total cost being the sum of one-hop costs can be justified for the lone packet model since when a packet is being forwarded there is no other packet transmission taking place). Our objective is to obtain placement policies $\pi$ to minimize a linear combination of the average total cost $\mathbb{E}_\pi C$ (expectation is with respect to the random evolution of the corridor) and the number of relays, $\mathbb{E}_\pi N$, used during the entire operation. Next we will describe the two different propagation models namely, the line-of-sight (LOS) and the non-line-of-sight (NLOS) models, along with the associated stochastic models for the path evolution. The optimal relay placement strategies for these two models are studied in detail in Sections 3 and 4, respectively.

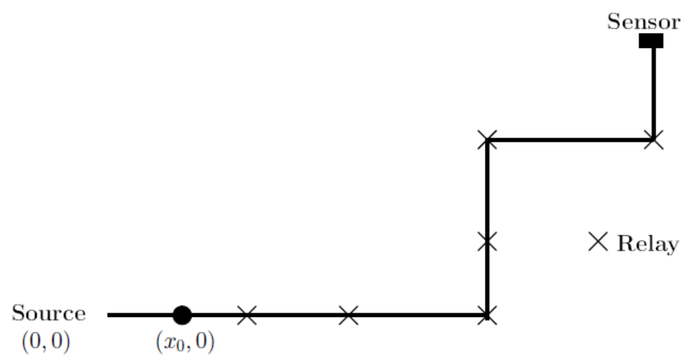



Figure 2.1: Mathematical abstraction of relay deployment along a random lattice path with radio-opaque walls.

1) *Line-of-Sight (LOS) propagation*: In this model a packet exchange can take place between two sucessive relays only if they are placed on the same straight line segment of the lattice path (e.g., within the same gallery of a building corridor). Thus, to establish communication, as shown in Fig. 1.4, a relay has to be invariably placed at all vertices where the path takes a turn. This model is best suited for deployment within a building where the walls are radio opaque. The stochastic path evolution model is as follows. From a vertex that the path has reached (without having terminated), the path will end at the next step with probability $p$. If the path does not terminate (with probability $1-p$), independently, at this point there is a turn with probability $q$, or there is no turn with probability $(1-q)$. Thus the probability that the path will continue with and without taking turns are $q(1-p)$ and $(1-q)(1-p)$, respectively. It is assumed that the person deploying the relays knows the number of steps that he has taken along the path since last placing a relay.

2) *Non-Line-of-Sight (NLOS) Propagation*: In this case, packet exchange can take place between any two successive relays even if they are not on the same straight line segment. In the building context this would correspond to the walls being radio transparent. The model is suitable when the deployment region is a thickly wooded forest where the deployment person is restricted to move only along some narrow path (lattice edges in our model). When the deployment person has reached some vertex, the path continues for one more step and terminates with probability $p$, or continues with probability $1-p$. In either case, the next step is Eastward with probability $q$ and Northward with probability $(1-q)$. In this version, the person deploying the relays is assumed to know the number of steps taken in the x direction and in y direction.

In our present work, the path loss model is assumed to be the same along all permitted links in the two versions. This does not conform well to the building setting, where propagation along corridors would have a different path loss model as compared to that through the walls, but would be applicable to an impromptu deployment as one

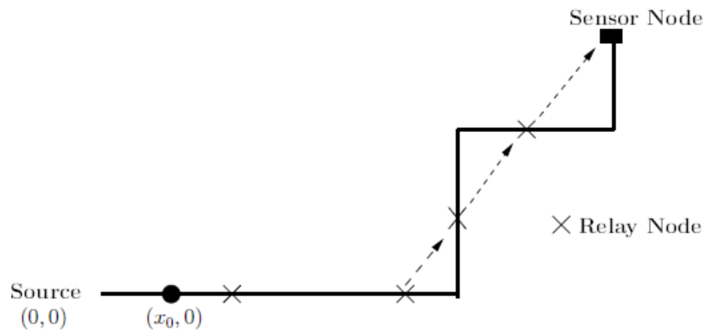



Figure 2.2: Mathematical abstraction of relay deployment along a corridor with radio-transparent walls

walks along a lattice in densely wooded area.

## 2.2 Deployment Policy $\pi$ and Problem Formulation

A deployment policy $\pi$ is a sequence of mappings $\mu_k : k \geq 0$, where at the $k$-th vertex (also referred to as the $k$-th step) of the path (provided that the path has not ended thus far) $\mu_k$ allows the deployment person to decide whether to *place* or *not to place* a relay where, in general, randomization over these two actions is allowed. The decision is based on the entire information available to the deployment person at the $k$-th step, namely the set of vertices traced by the path and the location of the previous vertices where relays were placed. Let $\Pi$ represent the set of all policies. For a given policy $\pi \in \Pi$, let $\mathbb{E}_\pi$ represent the expectation operator conditioned under using policy $\pi$. Let $C$ denote the total cost incurred and $N$ the total number of relays used. We are interested in solving the following problem,

$$\min_{\pi \in \Pi} \quad \mathbb{E}_\pi C + \lambda \mathbb{E}_\pi N \tag{2.1}$$

where $\lambda > 0$ may be interpreted as relay cost. Solving the problem in (2.1) can also help us solve the following constrained problem,

$$\begin{aligned} \min_{\pi \in \Pi} \quad & \mathbb{E}_\pi C \\ \text{Subject to:} \quad & \mathbb{E}_\pi N \leq \rho_{avg} \end{aligned} \tag{2.2}$$

where $\rho_{avg} > 0$ is a contraint on the average number of relays, by utilizing the following standard result:

**Lemma 2.2.1.** *Let $\pi_\lambda^* \in \Pi$ be an optimal policy for the unconstrained problem (2.1) such that $\mathbb{E}_{\pi_\lambda^*} N = \rho_{avg}$. Then*

*$\pi_\lambda^*$ is also optimal for the constrained problem (2.2).*

*Proof.* By the hypothesis about $\pi_\lambda^*$, we have for all $\pi \in \Pi$

$$\begin{aligned} \mathbb{E}_{\pi_\lambda^*} C + \lambda \mathbb{E}_{\pi_\lambda^*} N &\leq \mathbb{E}_\pi C + \lambda \mathbb{E}_\pi N \\ \implies \mathbb{E}_{\pi_\lambda^*} C &\leq \mathbb{E}_\pi C + \lambda(\mathbb{E}_\pi N - \rho_{avg}) \end{aligned}$$

But we have for all admissible $\pi$, $\mathbb{E}_\pi N \leq \rho_{avg}$. Hence we conclude that,

$$\mathbb{E}_{\pi_\lambda^*} C \leq \mathbb{E}_\pi C \quad \forall \pi \in \Pi \tag{2.3}$$

Hence it follows that $\pi_\lambda^*$ is optimal for the constrained problem (2.2) as well. ■

The specific details about the connection of the constrained problem to the "relaxed" problem is deferred to the subsequent chapters in their particular context. In Sections 3 and 4 (corresponding to LOS and NLOS propagation models, respectively) we focus on solving the problem in (2.1). We end this section by imposing few technical conditions on the one-hop cost function $d(\cdot)$.

### 2.2.1 Conditions on $d(\cdot)$:

$d(\cdot)$ satisfies the following conditions, (**C1**) $d(0) > 0$, (**C2**) $d(r)$ is convex and increasing in $r$, and (**C3**) for any $r$ and $\delta > 0$ the difference, $d(r+\delta) - d(r)$ increases to $\infty$.

(**C1**) is imposed considering the fact that it requires a non-zero amount of delay or power for transmitting a packet between two nodes, however close they may be. (**C2**) is a natural property of any cost function in general and (**C3**) is required in some of our proofs. These conditions are satisfied for the power cost, $P_m + \gamma r^\eta$, used in our numerical work, and in general by any reasonable cost function.

In the NLOS case, we will denote the one-hop cost between the locations $(0,0)$ and $(x,y) \in \Re^2$ as $d(x,y)$. We allow a general cost-function $d(x,y)$ endowed with the following properties, (**C4**) The function $d(x,y)$ is positive, twice continuously partially differentiable in variables $x$ and $y$ and

$$d_{xx}(x,y) > 0, \;\; d_{xy}(x,y) > 0, \;\; d_{yy}(x,y) > 0 \qquad\qquad \forall x,y \in \mathbb{R}_+ \tag{2.4}$$

Finally define, for $(m,n) \in \mathbb{Z}_+^2$, $\Delta_1(m,n) = d(m+1,n) - d(m,n)$ and $\Delta_2(m,n) = d(m,n+1) - d(m,n)$, where both $m$ and $n$ are measured in units of $\delta$. We now prove the following lemma.

**Lemma 2.2.2.** *$\Delta_1(m,n)$ and $\Delta_2(m,n)$ are non-decreasing in both the coordinates $m$ and $n$.*

*Proof.* It is easier to prove the lemma allowing the arguments $m$ and $n$ take values from the real line. We have,

$$\Delta_1(x,y) = d(x+\delta, y) - d(x,y)$$

Partially differentiating both sides w.r.t. $x$, we get

$$\begin{aligned}
\frac{\partial \Delta_1(x,y)}{\partial x} &= d_x(x+\delta,y) - d_x(x,y) && (2.5)\\
&= \delta d_{xx}(\zeta,y) \qquad x<\zeta<x+\delta && (2.6)\\
&> 0 && (2.7)
\end{aligned}$$

Where the equality in Eqn 2.6 follows from the application of Lagrange's Mean Value Theorem to the function $d_x(.,y)$ and the inequality in 2.7 follows from assumption 2.4.

The above proves the fact that $\Delta_1(x,y)$ is non-decreasing in $x$. To prove that $\Delta_1(x,y)$ is non-decreasing in $y$, we partially differentiate $\Delta_1(x,y)$ w.r.t. $y$ and obtain

$$\begin{aligned}
\frac{\partial \Delta_1(x,y)}{\partial y} &= d_y(x+\delta,y) - d_y(x,y) && (2.8)\\
&= \delta d_{xy}(\eta,y) \qquad x<\eta<x+\delta && (2.9)\\
&> 0 && (2.10)
\end{aligned}$$

Where the equality in Eqn 2.9 follows from the application of Lagrange's Mean Value Theorem to the function $d_y(.,y)$ and the inequality in 2.10 follows from assumption 2.4.

This shows that the function $\Delta_1(x,y)$ is non-decreasing in both the coordinates $x$ and $y$. In a similar way it can also be shown that $\Delta_2(x,y)$ is non-decreasing in $x$ and $y$ under the assumption 2.4. This completes the proof. ■

Now we consider a special case for the above cost-function $d(x,y)$ where the function depends only on the 2-norm of its arguments. This cost-function is realistic for deployment in a forest environment, where the cost-function is isotropic.

**Lemma 2.2.3.** *$d(x,y)$ is convex in $(x,y)$, where $(x,y) \in \mathbb{R}^2$*

*Proof.* To show this, we use the composition rule. Define the function $g(x,y) = \sqrt{x^2+y^2}$ which is nothing but the 2-norm of the vector $(x,y)^T$.Let us denote this vector by $\boldsymbol{r}$.

Then we have, via triangle inequality of the 2-norm, for $0 \le \lambda \le 1$,

$$|\lambda \boldsymbol{r}_1 + (1-\lambda)\boldsymbol{r}_2|_2 \le \lambda|\boldsymbol{r}_1|_2 + (1-\lambda)|\boldsymbol{r}_2|_2 \qquad (2.11)$$

This shows that the function $g(x,y) \equiv \sqrt{x^2+y^2}$ is convex in $(x,y)$. Also the delay function $d(.)$ is assumed to be convex and non-decreasing in its argument. Hence by using the composition rule 3.10 of ([6]), we conclude that the function $d(x,y) \equiv d(\sqrt{x^2+y^2})$ is convex in $(x,y) \in \mathbb{R}^2$. ■

Now we make the following *assumption* on the function $d(x,y)$.

- $d(x,y)$ is a convex increasing function of $(x^2+y^2)$, i.e., $d(x,y)$ may be written as

$$d(x,y) = h(x^2+y^2) \tag{2.12}$$

Where $h(x)$ is convex and increasing in $x$.

This assumption may be justified for the most common delay function as considered in [13]. From Chapter 3 of [13], we observe that using lone-packet model, the average packet transmission delay between the transmitter and receiver separated by a distance $r$ is given by $d_{\eta,u}(r) = \frac{T}{(1-e^{-\beta u r^{-\eta}})^l}$, where $u$ captures the contribution of fading, $\eta$ is the path-loss exponent, $\beta$ and $T$ and $l$ are constants depending on the protocol used and on the fading distribution and are independent of $r$.

It was shown in Lemma 3.1 of [13] that for $\eta > 1$, $d_{\eta,u}(r)$ is strictly convex, strictly increasing in $r$. Moreover $d_{\eta,u}(r)$ its derivative w.r.t. $r$ increases to infinity as $r \to \infty$.

For the above delay function, we have

$$\begin{aligned} d_{\eta,u}(r) &= \frac{T}{(1-e^{-\beta u r^{2(-\frac{\eta}{2})}})^l} && (2.13)\\ &= d_{\frac{\eta}{2},u}(r^2) && (2.14) \end{aligned}$$

Hence by the same argument as in [13], we conclude that for $\frac{\eta}{2} > 1$, i.e., $\eta > 2$, $d_{\eta,u}(r)$ is a convex increasing function of $r^2$, and $d_{\eta,u}(r)$ and its derivative w.r.t $r^2$ goes to infinity as $r \to \infty$. Averaging over the distribution of $U$, we obtain that $d(r) \equiv \mathbb{E}_u d_{\eta,u}(r)$ is convex and increasing in $r^2$ with its first derivative going to $\infty$ as $r \to \infty$ for $\eta > 2$.

The assumption may be also justified similarly for the most common one-hop power function as considered in [14]. There we have one-hop power function $P_\eta(r) = P_m + \gamma r^\eta$, where $P_m, \gamma, \eta$ are constants. Similar conclusion may be drawn for $P_\eta(r^2)$ by noting that

$$P_\eta(r) = P_{\frac{\eta}{2}}(r^2) \tag{2.15}$$

It is easy to check that the conditions **(C1), (C2), (C3)** and **(C4)** are satisfied for this function.

# Chapter 3

# The Line-Of-Sight (LOS) Case

Recall that in the LOS case, we are forced to place a relay at each turn in the random lattice path. In this section, we show that, due to this feature, the problem can be decomposed into a random number of problems each being that of deploying relays on a straight line path without turns. The problem on a straight line can be formulated as an optimal stopping problem whose solution is a threshold policy. The problem on a straight line was considered by Mondal et al. [13]. We show that the policy obtained therein can be viewed as a one-step-look-ahead (OSLA) rule.

## 3.1 MDP Formulation

We formulate the problem as a Markov Decision Process (MDP). The decision of placing or not placing a relay is taken at each step $k \in \mathbb{Z}_+$ of the path (with $k = 0$ corresponding to $(0,0)$). We consider the random process $S_k = (X_k, Z_k)$, where $X_k$ is the number of steps between location $k$ and the last node (relay or control centre), and $Z_k \in \{\mathsf{e}, \mathsf{t}, \mathsf{c}\}$. $Z_k = \mathsf{e}$ means that at step $k$ the random lattice path has ended, $Z_k = \mathsf{t}$ means that the path takes a turn, and $Z_k = \mathsf{c}$ means that the path will continue in the same direction for at least one more step. The state-space is thus given by:

$$\mathscr{S} = \{(x,z) : x \in \mathbb{Z}_+, z \in \{\mathsf{e}, \mathsf{t}, \mathsf{c}\}\} \cup \{\phi\} \tag{3.1}$$

where $\{\phi\}$ denotes the cost-free terminal state, i.e., the state after the end of the path has been discovered. The action taken at step $k$ is denoted $U_k \in \{0,1\}$, where $U_k = 1$ is the action to place a relay, and $U_k = 0$ is the action of not placing a relay. The permissible action sets are as follows

- $\mathbf{A}_{(x,z)} = \{0,1\}$ for $z = \mathsf{c}$
- $\mathbf{A}_{(x,z)} = \{1\}$ for $z = \{\mathsf{t}, \mathsf{e}\}$

Finally, by the sequence $U_k \in \mathbb{A}_k, k \geq 0$, we denote the sequence of actions. Given that $(X_k, Z_k) = (x, z)$, and if the action $U_k$ is taken, we now write down the state transition probabilities.

- If $u_k$ is 0,

$$(x, \mathsf{c}) \longrightarrow \quad (x+1, \mathsf{c}) \quad \text{w.p.} \quad (1-p)(1-q) \tag{3.2}$$

$$(x, \mathsf{c}) \longrightarrow \quad (x+1, \mathsf{t}) \quad \text{w.p.} \quad (1-p)q \tag{3.3}$$

$$(x, \mathsf{c}) \longrightarrow \quad (x+1, \mathsf{e}) \quad \text{w.p.} \quad p \tag{3.4}$$

- If $u_k$ is 1,

$$(x, \mathsf{c}) \longrightarrow \quad (1, \mathsf{c}) \quad \text{w.p.} \quad (1-p)(1-q) \tag{3.5}$$

$$(x, \mathsf{c}) \longrightarrow \quad (1, \mathsf{t}) \quad \text{w.p.} \quad (1-p)q \tag{3.6}$$

$$(x, \mathsf{c}) \longrightarrow \quad (1, \mathsf{e}) \quad \text{w.p.} \quad p \tag{3.7}$$

- If $Z_k = \mathsf{t}$, we have only one allowable action, i.e., $u_k = 1$. In this case the state-transitions are given as follows

$$(x, \mathsf{t}) \quad \longrightarrow \quad (1, \mathsf{c}) \quad \text{w.p.} \quad (1-p)(1-q) \tag{3.8}$$

$$(x, \mathsf{t}) \quad \longrightarrow \quad (1, \mathsf{t}) \quad \text{w.p.} \quad (1-p)q \tag{3.9}$$

$$(x, \mathsf{t}) \quad \longrightarrow \quad (1, \mathsf{e}) \quad \text{w.p.} \quad p \tag{3.10}$$

- If $Z_k = \mathsf{e}$, the allowable action is $u_k = 1$ and we enter a cost free terminal state $\phi$.

$$(x, \mathsf{e}) \longrightarrow \{\phi\} \quad \text{w.p.} \quad 1 \tag{3.11}$$

- If $S_k = \phi$, we return to the same terminal state $\phi$ w.p. 1, irrespective of the control $u_k$

$$\{\phi\} \longrightarrow \{\phi\} \quad \text{w.p.} \quad 1 \tag{3.12}$$

Finally, if the state is $\phi$, we remain in $\phi$ thereafter.

For $s \in \mathscr{S}$, the one step cost is given by:

$$c(s,u) = \begin{cases} d(x) & \text{if } s = (x, \mathsf{e}) \\ \lambda + d(x) & \text{if } u = 1 \text{ and } s = (x, \mathsf{c}) \\ 0 & \text{if } u = 0 \text{ or } s = \phi. \end{cases} \tag{3.13}$$

A policy $\pi$ is a sequence of mappings $\{\mu_k\}_{k\geq 0}$, where $\mu_k$ maps a state $S_k$ to the corresponding set of feasible actions. For a given policy $\pi$, the expectation of the total cost $\sum_{k=0}^{\infty} c(S_k, \mu_k(S_k))$ is equal to the objective of the problem in (2.1)

## 3.2 Solving the Relaxed Problem

### 3.2.1 Optimal Value and Bellman Equation

We have a total cost infinite horizon MDP with countable state space, finite action sets and non-negative costs per stage. Let $J_\lambda(x,z)$ denote the optimal cost-to-go at state $(x,z)$ and define $v=(1-p)q$ the probability of a turn at the next step without the lattice path ending. By [5, Prop. 1.1, Page 137] $J_\lambda(\cdot,\cdot)$ is a solution to Bellman's equation. It follows that

$$J_\lambda(x,\mathtt{t}) \quad = \quad \lambda + d(x) + vJ_\lambda(1,\mathtt{t}) + pd(1) + (1-p-v)J_\lambda(1,\mathtt{c}) \tag{3.14}$$

which may be understood as follows. When at state $(x,\mathtt{t})$ a relay must be place, thus incurring a cost of $\lambda + d(x)$. The next state is $(1,\mathtt{t})$ w.p. $v$, the cost-to-go from which is $J_\lambda(1,\mathtt{t})$; $(1,\mathtt{e})$ w.p. $p$, the cost-to-go from which is $d(1)$; $(1,\mathtt{c})$ w.p. $(1-p-v)$, the cost-to-go from which is $J_\lambda(1,\mathtt{c})$.

When the state is $(x,\mathtt{c})$, both actions are possible. Let $c_p(x)$ and $c_{np}(x)$ be the average cost of placing and not placing respectively, the expressions for which are given by $c_p(x) = \lambda + d(x) + vJ_\lambda(1,\mathtt{t}) + pd(1) + (1-p-v)J_\lambda(1,\mathtt{c})$ and $c_{np}(x) = vJ_\lambda(x+1,\mathtt{t}) + pd(x+1) + (1-p-v)J_\lambda(x+1,\mathtt{c})$. Bellman's equation then yields

$$J_\lambda(x,\mathtt{c}) \quad = \quad \min\{c_p(x), c_{np}(x)\}. \tag{3.15}$$

Define the *optimal placement set* $\mathscr{P}_\lambda$ as the set of all $x$ such that in states of the form $\{x,\mathtt{c}\}$ the optimal action is to place a relay. Using (3.14) to (3.15) and [5, Prop. 1.3, Page 143], the placement set can be written as

$$\begin{aligned}\mathscr{P}_\lambda &= \{x \in \mathbb{Z}_+ : c_p(x) \leq c_{np}(x)\} \\ &= \Big\{x \in \mathbb{Z}_+ : \lambda + (p+v)d(1) + (1-p-v)J_\lambda(1,\mathtt{c}) \leq d(x+1) - d(x) + (1-p-v)(J_\lambda(x+1,\mathtt{c}) - d(x+1))\Big\}\end{aligned} \tag{3.16}$$

Let us now consider the problem of placing relays along the straight line segment between two consecutive turns in the lattice path. We will show that the solution to this problem is also optimal for the problem of placing relays on the random lattice path with LOS propagation. Notice that the probability that such a line segment ends at the next step is $(p+v)$. The state space for this subproblem is $\mathscr{S}_g = \{(x,z) : x \in \mathbb{Z}_+, z \in \{\mathtt{c},\mathtt{e}\}\} \cup \{\phi\}$. Let $G_\lambda(x,z)$ be the optimal cost-to-go for this subproblem when the state is $(x,z)$. As before, we obtain Bellman's equation and an optimal placement set $\hat{\mathscr{P}}_\lambda$ as

$$G_\lambda(x,\mathsf{c}) = \min\left\{\lambda + d(x) + (p+v)d(1) + (1-p-v)G_\lambda(1,\mathsf{c}), (p+v)d(x+1) + (1-p-v)G_\lambda(x+1,\mathsf{c})\right\}, \tag{3.17}$$

$$\hat{\mathscr{P}}_\lambda = \left\{x \in \mathbb{Z}_+ : \lambda + (p+v)d(1) + (1-p-v)G_\lambda(1,\mathsf{c}) \leq d(x+1) - d(x) + (1-p-v)(G_\lambda(x+1) - d(x+1))\right\} \tag{3.18}$$

The following result asserts that optimal placement policies for the straight line segments in the lattice path (in the LOS case) are threshold policies.

**Lemma 3.2.1.** *There exists a threshold $r_\lambda^*$ such that $\hat{\mathscr{P}}_\lambda = \{r_\lambda^*, r_\lambda^*+1, \cdots, \infty\}$.*

*Proof.* The proof was provided my Mondal et al. (see [13]) and is based on a monotonicity result that follows from a value iteration argument. ∎

**Theorem 3.2.2.** $\mathscr{P}_\lambda = \hat{\mathscr{P}}_\lambda$

*Proof.* Combining Equations (3.15) and (3.14), we see that

$$\begin{aligned} J_\lambda(x,\mathsf{c}) \quad = \quad & \min\{\frac{1}{1-v}\lambda + d(x) + \frac{p+v}{1-v}d(1) + \frac{1-p-v}{1-v}J_\lambda(1), \frac{v}{1-v}\lambda + qd(x+1) + \frac{v}{1-v}(p+v)d(1) + \\ & \frac{v}{1-v}(1-p-v)J_\lambda(1) + pd(x+1) + (1-p-v)J_\lambda(x+1)\} \end{aligned} \tag{3.19}$$

Now we define the function $Q(x)$ by

$$J_\lambda(x,\mathsf{c}) = Q(x) + C \qquad x \in \mathbb{Z}_+ \tag{3.20}$$

where,

$$C = \frac{v}{1-v}\lambda + \frac{v}{1-v}(p+v)d(1) + (1-p-v)\left(\frac{J_\lambda(1)}{1-v} - Q(1)\right) \tag{3.21}$$

And,

$$J_\lambda(1,\mathsf{c}) - Q(1) = C \tag{3.22}$$

The above two equations yield the following value for the constant $C$

$$C = \frac{v}{1-v}\left(\frac{1-p-v}{p+v}J_\lambda(1) + d(1) + \frac{\lambda}{p+v}\right) \tag{3.23}$$

Substituting $J_\lambda(x,\mathsf{c})$ in terms of $Q(x)$ in Eqn (3.19), we obtain

$$Q(x) = \min\{\lambda + d(x) + (p+v)d(1) + (1-p-v)Q(1), (p+v)d(x+1) + (1-p-v)Q(x+1)\} \tag{3.24}$$

The above step follows by using Eqns (3.21) and (3.22) and simple algebraic manipulations. But we note that the functional equations describing $G_\lambda(x,\mathsf{c})$ (3.17) and $Q(x)$ (3.24) are identical. In general, these Bellman Equations will have infinitely many solutions for the undiscounted problem considered above. However, from Proposition 1.2 of [5], we know that the optimal cost-to-go is the minimal solution of the Bellman Equations, which is unique. Hence, we conclude that $Q(x) = G_\lambda(x,\mathsf{c}), \forall x \in \mathbb{Z}_+$. This in turn implies that,

$$J_\lambda(x,\mathsf{c}) = G_\lambda(x,\mathsf{c}) + C \tag{3.25}$$

With $C$ given as above.

Now consider the set $\mathscr{P}_\lambda$, which is given as follows

$$\begin{aligned}\mathscr{P}_\lambda &= \{x \in \mathbb{Z}_+ : c_p(x) \le c_{np}(x)\}\\ &= \{x \in \mathbb{Z}_+ : \lambda + (p+v)d(1) + (1-p-v)J_\lambda(1) \le d(x+1) - d(x) + (1-p-v)(J_\lambda(x+1) - d(x+1))\}\end{aligned}$$

Now we substitute $J_\lambda(x,\mathsf{c}) = G_\lambda(x,\mathsf{c}) + C, \quad \forall x \in \mathscr{P}_\lambda$. This yields,

$$\begin{aligned}\mathscr{P}_\lambda &= \{x \in \mathbb{Z}_+ : \lambda + (p+v)d(1) + (1-p-v)G_\lambda(1,\mathsf{c}) \le d(x+1) - d(x) + (1-p-v)(G_\lambda(x+1,\mathsf{c}) - d(x+1))\}\\ &= \hat{\mathscr{P}}_\lambda \end{aligned} \tag{3.26}$$

Where the last equality follows from Eqn. (3.18). ■

**Proposition 3.2.3.** $G_\lambda(0) = (p+v)d(1) + (1-p-v)G_\lambda(1)$

*Proof.* Keeping in mind Eqn. (3.17), we need to prove that, at the state $(0,\mathsf{c})$, it is optimal not to place a relay and move on to the next step. Note that if it *had been* optimal to place at the state $(0,\mathsf{c})$, at the next step, we return to the same state, viz., $(0,\mathsf{c})$. Now, because of the stationarity of the optimal policy, we would keep placing the relays at the same point, and since "relay-cost" $\lambda > 0$ and $d(0) > 0$, the expected cost for this policy is $\infty$. Hence it is optimal *not to place* a relay node at the state $(0,\mathsf{c})$, and from equation (3.17), we have $G_\lambda(0) = (p+v)d(1) + (1-p-v)G_\lambda(1)$. ■

**Proposition 3.2.4.** *For the case of the corridor with opaque walls, the optimal cost-to-go $J_\lambda(x)$ is related to the optimal cost-to-go for the single gallery problem, $G_\lambda(x)$, as $\forall x \in \mathbb{Z}_+$*

$$J_\lambda(x) = G_\lambda(x) + \frac{v}{p}(\lambda + G_\lambda(0)) \tag{3.27}$$

*Proof.* In Theorem 3.2.2, we have shown that

$$J_\lambda(x) = G_\lambda(x) + C \qquad \forall x \in \mathbb{Z}_+ \tag{3.28}$$

with

$$C = \frac{v}{1-v}\left(\frac{1-p-v}{p+v}J_\lambda(1) + d(1) + \frac{\lambda}{p+v}\right) \tag{3.29}$$

From proposition (3.2.3), we have

$$G_\lambda(0) = (p+v)d(1) + (1-p-v)G_\lambda(1)$$

i.e.,

$$\frac{G_\lambda(0)}{p+v} = d(1) + \frac{(1-p-v)G_\lambda(1)}{p+v} \tag{3.30}$$

Now we refer to Eqn.(3.29) and substitute $J_\lambda(1) = G_\lambda(1) + C$. This yields,

$$C = \frac{v}{1-v}(\frac{1-p-v}{p+v}G_\lambda(1) + d(1) + \frac{\lambda}{p+v}) + C\frac{r(1-p-v)}{(1-v)(p+v)}$$

i.e.,

$$C = \frac{v}{p}(\lambda + G_\lambda(0)) \tag{3.31}$$

Where we have utilized (3.30).

Combining equations (3.27) and (3.31), we have $\forall x \in \mathbb{Z}_+$

$$J_\lambda(x) = G_\lambda(x) + \frac{v}{p}(\lambda + G_\lambda(0)) \tag{3.32}$$

■

**Remark:** Here is a renewal argument that yields the same result. Notice from the Bellman Equation for $G_\lambda(x)$ that we do not account for the cost of the node placed when the gallery ends and the corridor takes a turn. The renewal argument then yields:

$$J_\lambda(x) = G_\lambda(x) + \frac{v}{r+p}(\lambda + J_\lambda(0)) \tag{3.33}$$

where $\frac{v}{r+p}$ is the probability that the corridor continues given that a gallery ends. We do not account for the cost of the sensor placed at the end of the corridor. This is a constant cost that is incurred for all deployments. The same argument also shows that,

$$J_\lambda(0) = G_\lambda(0) + \frac{v}{r+p}(\lambda + J_\lambda(0)) \tag{3.34}$$

from which it follows that

$$J_\lambda(0) = \frac{r+p}{p} G_\lambda(0) + \frac{v}{p}\lambda$$

i.e.,

$$J_\lambda(0) = G_\lambda(0) + \frac{v}{p}(G_\lambda(0) + \lambda) \tag{3.35}$$

Substituting into the expression for $J_\lambda(x)$ yields

$$J_\lambda(x) = G_\lambda(x) + \frac{v}{r+p}\left((\lambda + G_\lambda(0)) + \frac{v}{p}(G_\lambda(0) + \lambda)\right) \tag{3.36}$$

i.e.,

$$J_\lambda(x) = G_\lambda(x) + \frac{v}{p}(G_\lambda(0) + \lambda) \tag{3.37}$$

We have already proved in Theorem 3.2.2 that the structure of the placement set $\mathscr{P}_\lambda$ for the multiple corridor problem is identical with that of the single corridor problem which was studied in the thesis ([13]) and in the paper ([14]). It is shown in ([13]) that the optimal placement set $\mathscr{P}_\lambda$, for the straight-line corridor problem, is of the form

$$\mathscr{P}_\lambda = [r_\lambda^*, r_\lambda^* + 1, r_\lambda^* + 2, \ldots, \infty) \tag{3.38}$$

for an optimal $r_\lambda^* \in Z_+$. This implies that in our problem of a corridor with turns, the optimal placement policy will also have threshold form. Now we formulate the problem as an *Optimal Stopping Problem* in the next section.

### 3.2.2 Optimal Stopping Formulation

In the previous section we obtained the structure of an optimal placement policy. In this section, we will show that the one-step-look-ahead (OSLA) rule provides an equivalent characterisation of optimal placement sets. This result then yields a simple computational procedure for $r_\lambda^*$. Consider the placement problem along a straight line segment. The motivation to consider the OSLA rule comes from the observation that when a relay is placed and the path continues, it is a *renewal point* for the decision process. The cost-to-go at this point (i.e., $G_\lambda(0,\mathrm{c})$) can be taken to be the termination cost in a *stopping problem* that starts at the point at which the previous relay was placed. For this stopping problem, the OSLA policy compares at each step the cost of placing a relay with the cost of continuing for one more step and placing a relay. When placing a relay at distance $x$, we incur a cost of $d(x) + \lambda$ and the cost-to-go from which is $G_\lambda(0,\mathrm{c})$ since the placement point is a renewal point of the process. The cost of continuing one step ahead and placing is $d(x+1)$ if the gallery ends and $d(x+1)+\lambda+G_\lambda(0,\mathsf{c})$ otherwise. Thus, the associated placement set is given by

$$\begin{aligned}\overline{\mathscr{P}}_\lambda &= \Big\{x \in \mathbb{Z}_+ : d(x)+\lambda+G_\lambda(0,\mathsf{c}) \le (p+v)d(x+1)+(1-p-v)(d(x+1)+\lambda+G_\lambda(0,\mathsf{c}))\Big\} \\ &= \Big\{x \in \mathbb{Z}_+ : d(x+1)-d(x) \ge (p+v)(\lambda+G_\lambda(0,\mathsf{c}))\Big\}. \qquad (3.39)\end{aligned}$$

**Lemma 3.2.5.** *There exists a threshold* $\bar{r}_\lambda$ *such that* $\overline{\mathscr{P}}_\lambda = \{\bar{r}_\lambda, \bar{r}_\lambda+1, \cdots, \infty\}$.

*Proof.* This follows from the convexity property of the function $d(\cdot)$. The convexity of $d(\cdot)$ implies that $d(x+1)-d(x)$ is non-decreasing in $x$. We have also assumed that $d(x+1)-d(x)$ tends to infinity as $x$ increases (see Section 2.2.1). On the other hand, the right hand side (RHS) of (3.39) does not depend on $x$. This shows the existence of a threshold for the OSLA policy. ■

We now conclude this section by showing that the OSLA policy is optimal for relay placement along any straight line segment in the lattice path.

**Theorem 3.2.6.** $\hat{\mathscr{P}}_\lambda = \overline{\mathscr{P}}_\lambda$, *i.e.,* $r^*_\lambda = \bar{r}_\lambda$.

*Proof.* **A.** $\hat{\mathscr{P}}_\lambda \subset \bar{\mathscr{P}}_\lambda$ :

Consider any $x \in \hat{\mathscr{P}}_\lambda$. Since the optimal policy for all such $x$ is to place a relay, we may write, from Equation (3.17) that

$$G_\lambda(x,\mathsf{c}) = \lambda + d(x) + (p+v)d(1) + (1-p-v)G_\lambda(1,\mathsf{c}) \qquad (3.40)$$

Also if $x \in \hat{\mathscr{P}}_\lambda$, then we must have $x+1 \in \hat{\mathscr{P}}_\lambda$ (because of the threshold structure of the optimal policy). Hence we also have

$$G_\lambda(x+1,\mathsf{c}) = \lambda + d(x+1) + (p+v)d(1) + (1-p-v)G_\lambda(1,\mathsf{c}) \qquad (3.41)$$

Using Eqn. (3.41) to substitute for $G_\lambda(x+1,\mathsf{c})$, we obtain, from the definition of $\hat{\mathscr{P}}_\lambda$,

$$\begin{aligned}&\lambda + d(x) + (p+v)d(1) + (1-p-v)G_\lambda(1,\mathsf{c}) \le (p+v)d(x+1) + (1-p-v)(\lambda + d(x+1) + (p+v)d(1) + \\ &(1-p-v)G_\lambda(1,\mathsf{c})) \qquad (3.42)\\ &\text{i.e.} \quad d(x+1)-d(x) \ge (p+v)(\lambda + (p+v)d(1) + (1-p-v)G_\lambda(1,\mathsf{c})) \qquad (3.43)\end{aligned}$$

Now, as asserted in Proposition (3.2.3),

$$G_\lambda(0,\mathsf{c}) = (p+v)d(1) + (1-p-v)G_\lambda(1,\mathsf{c}) \qquad (3.44)$$

substituting (3.44) into (3.42), we obtain

$$d(x+1)-d(x) \geq (p+v)(\lambda + G_\lambda(0,\mathsf{c})) \tag{3.45}$$

i.e. $x \in \bar{\mathscr{P}}_\lambda$. Which proves

$$\mathscr{P}_\lambda \subset \bar{\mathscr{P}}_\lambda \tag{3.46}$$

Taking account the threshold structure of the sets, this in turn implies that

$$r^*_\lambda \geq \bar{r}_\lambda \tag{3.47}$$

To show $\hat{\mathscr{P}}_\lambda = \bar{\mathscr{P}}_\lambda$ as claimed in the theorem, it is now enough to prove that $r^*_\lambda = \bar{r}_\lambda$.
**B.** $r^*_\lambda = \bar{r}_\lambda$ :

We have already shown that $r^*_\lambda \geq \bar{r}_\lambda$. If possible, suppose that $r^*_\lambda > r^*_\lambda - 1 \geq \bar{r}_\lambda$. Then we have $r^*_\lambda - 1 \in \bar{P}_\lambda$, but $r^*_\lambda - 1 \notin \hat{\mathscr{P}}_\lambda$ and $r^*_\lambda \in \hat{\mathscr{P}}_\lambda$.
Since $r^*_\lambda - 1 \in \bar{\mathscr{P}}_\lambda$, we have

$$d(r^*_\lambda) - d(r^*_\lambda - 1) \geq (p+v)(\lambda + G_\lambda(0,\mathsf{c})) \tag{3.48}$$

We work backward in the above derivation up to equation (3.42) and utilize Eqn. (3.44) to obtain

$$\lambda + d(r^*_\lambda - 1) + G_\lambda(0,\mathsf{c}) \leq (p+v)d(r^*_\lambda) + (1-p-v)(\lambda + d(r^*_\lambda) + G_\lambda(0,\mathsf{c})) \tag{3.49}$$

Since, $r^*_\lambda - 1 \notin \hat{\mathscr{P}}_\lambda$, from Eqn. (3.17), we have

$$(p+v)d(r^*_\lambda) + (1-p-v)G_\lambda(r^*_\lambda,\mathsf{c}) < \lambda + d(r^*_\lambda - 1) + G_\lambda(0,\mathsf{c}) \tag{3.50}$$

Also since $r^*_\lambda \in \hat{\mathscr{P}}_\lambda$, we have from Eqn. (3.17),

$$G_\lambda(r^*_\lambda,\mathsf{c}) = \lambda + d(r^*_\lambda) + G_\lambda(0,\mathsf{c}) \tag{3.51}$$

Combining Equations (3.50) and (3.51), we see that

$$(p+v)d(r^*_\lambda) + (1-p-v)(\lambda + d(r^*_\lambda) + G_\lambda(0,\mathsf{c})) < \lambda + d(r^*_\lambda - 1) + G_\lambda(0,\mathsf{c}) \tag{3.52}$$

Now we have a contradiction between Eqns. (3.49) and (3.52). Hence, what we assumed at the outset was wrong and we must have $r^*_\lambda = \bar{r}_\lambda$. This proves the theorem. ■

## 3.3 Computation of the Optimal Threshold $r^*_\lambda$

In this section, we leverage the result in Theorem 3.2.6 to provide a simple method for computing the optimal placement thresholds $r^*_\lambda$. For the straight line problem and for a given threshold $r$, let us first define $g_\lambda(r)$ to be the expected cost-to-go starting with the state $(0,\mathsf{c})$ (i.e., just after a relay has been placed). Using a renewal argument, it can be seen that

$$g_\lambda(r) = \sum_{k=1}^{r}(1-(p+v))^{(k-1)}(p+v)d(k) + (1-(p+v))^r(\lambda + d(r) + g_\lambda(r)) \tag{3.53}$$

where the first term accounts for cases where the gallery ends at $k$ th step which is before or at $r$ (a cost of $d(k)$ is induced) and the second term for other cases (a relay is placed at $r$ and the process restarts). Solving for $g_\lambda(r)$, we get

$$g_\lambda(r) = \frac{1}{1-(1-p-v)^r}\left((1-p-v)^r(\lambda + d(r)) + \sum_{k=1}^{r}(p+v)(1-p-v)^{k-1}d(k)\right). \tag{3.54}$$

Clearly, $g_\lambda(r^*_\lambda) = G_\lambda(0,\mathsf{c})$. The following result provides a simple technique for computing $r^*_\lambda$.

**Proposition 3.3.1.** $r^*_\lambda = \min\{r \in \mathbb{Z}_+ : d(r+1) - d(r) \geq (p+v)(\lambda + g_\lambda(r))\}$.

*Proof.* Let $\Delta(r) = d(r+1) - d(r)$. Define the sets

$$\begin{aligned} S_\lambda &= \{r \in \mathbb{Z}_+ : \Delta(r) \geq (p+v)(\lambda + g_\lambda(r))\} \\ S^*_\lambda &= \{r \in \mathbb{Z}_+ : \Delta(r) \geq (p+v)(\lambda + g_\lambda(r^*_\lambda))\} \end{aligned}$$

Then, we have from the OSLA rule (3.39) $r^*_\lambda = \min S^*_\lambda$. Let $r_\lambda = \min S_\lambda$. We want to show that $r^*_\lambda = r_\lambda$. We note that $g_\lambda(r)$ is minimized at $r = r^*_\lambda$, i.e., $g_\lambda(r^*_\lambda) \leq g_\lambda(r), \ \forall r \in \mathbb{Z}_+$. Now since $r_\lambda \in S_\lambda$, we have:

$$\Delta(r_\lambda) \geq (p+v)(\lambda + g_\lambda(r_\lambda)) \geq (p+v)(\lambda + g_\lambda(r^*_\lambda))$$

This implies that $r_\lambda \in S^*_\lambda$ and hence, $r^*_\lambda \leq r_\lambda$. On the other hand, since $r^*_\lambda \in S^*_\lambda$, we have $\Delta(r^*_\lambda) \geq (p+v)(\lambda + g_\lambda(r^*_\lambda))$. This implies that, $r^*_\lambda \in S_\lambda$. Hence, $r_\lambda \leq r^*_\lambda$. ■

### 3.3.1 A bound on $g_\lambda(r)$

In this subsection, we find an upperbound of $g_\lambda(r)$. This enables us to restrict our search range to finite values while computing $r^*_\lambda$ using Proposition 3.3.1. Let

$$S(r)=\sum_{k=1}^{r}(p+v)(1-p-v)^{k-1} \tag{3.55}$$

Note that $S(r)$ is monotonically increasing in $r \in \mathbb{Z}_{+}$. Hence $S(1)=p+v \leq S(r)\ \forall r \in \mathbb{Z}_{+}$.

We have the following expression for $g_{\lambda}(r)$ for the LOS case.

$$g_{\lambda}(r)=\frac{(1-p-v)^{r}}{S(r)}(\lambda+d(r))+\frac{1}{S(r)}\sum_{k=1}^{r}(p+v)(1-p-v)^{k-1}d(k) \tag{3.56}$$

Now we upperbound each of the two terms seperately. First consider the second term.

$$\begin{aligned}
& \frac{1}{S(r)}\sum_{k=1}^{r}(p+v)(1-p-v)^{k-1}d(k) && (3.57)\\
\leq\ & \frac{1}{S(r)}\sum_{k=1}^{\infty}(p+v)(1-p-v)^{k-1}d(k) && (3.58)\\
\leq\ & \frac{1}{S(1)}\sum_{k=1}^{\infty}(p+v)(1-p-v)^{k-1}d(k) && (3.59)\\
=\ & \frac{1}{p+v}\mathbb{E}d && (3.60)
\end{aligned}$$

Here $\mathbb{E}()$ denotes the expectation over the geometric distribution $\{(p+v)(1-p-v)^{k-1}, k=1,2,\ldots,\}$.

Since the raw moments of all order ($\geq 1$) of the Geometric distribution is finite, this expectation can be shown to be finite, for $d(k)=P_m+\gamma k^{\eta}$, $\eta \geq 1$.

Now consider the first term.

$$\begin{aligned}
& \frac{(1-p-v)^{r}}{S(r)}(\lambda+d(r)) && (3.61)\\
\leq\ & \frac{(1-p-v)^{r}}{p+v}(\lambda+d(r)) && (3.62)\\
=\ & \frac{1}{p+v}\exp(-\beta r)(\lambda+d(r)) && (3.63)
\end{aligned}$$

Where, $\beta=-\ln(1-p-v)>0$.

We can calculate an explicit upperbound for the function $h(r)=\exp(-\beta r)(\lambda+d(r)), r \in \mathbb{R}$ , for $d(r)=P_m+\gamma r^{\eta}$ using simple calculus.

Thus we have a computable upperbound on $g_{\lambda}(r)$ for the type of delay functions considered in the thesis.

## 3.4 An Efficient Fixed Point Iteration Algorithm Using the OSLA rule

In this section, we present an efficient fixed point iteration algorithm using the OSLA rule as presented in 3.39 for finding the optimal policy as well as the optimal cost-to-go for a given set of parameter values $\lambda, p, q$ and the one-step delay function $d(\cdot)$. The advantage of this algorithm over the direct minimization of the cost-to-go function $g(\cdot)$ as discussed in the previous paragraphs is two folds.

- On the theoretical side, this iterative algorithm avoids performing explicit optimization altogether, which, otherwise is to be performed numerically over the whole range of $\mathbb{R}_+$. Without any structure on the objective function, direct numerical minimization of $g(\cdot)$ is difficult and often unsatisfactory, which invariably uses some sort of heuristic search over $\mathbb{R}_+$.
- On the practical side, this algorithm is observed to be extremely fast to converge (requires $3-4$ iterations typically) and is proved to be convergent within a finite number of iterations. This is extremely desirable in applications such as rapid deployment in emergency situations for which the algorithm is targetted to.

In the following, we first present the iterative algorithm. Then we prove its correctness and finite termination properties.

For the sake of brevity and clarity of expressions, we consider the equivalent St. Line problem with parameter $p$. We also replace $g_\lambda(\cdot)$ by $g(\cdot)$ and $r^*_\lambda$ by $r^*$ and $J_\lambda(0) = \min g(\cdot)$ by $g^*$ in this section.

**Algorithm 1** Computes $g^*$ and $r^*$ for the St. Line Problem

**Require:** $0 < p < 1$, $\lambda \geq 0$

$h \leftarrow 0$

**while** 1 **do**

$$r_h \leftarrow \min\{r \in \mathbb{Z}_+ : p(\lambda + h) \leq \Delta(r)\}$$

$$g_h \leftarrow \frac{1}{1-(1-p)^{r_h}} \left( (1-p)^{r_h}(\lambda + d(r_h)) + \sum_{k=1}^{r_h} (p)(1-p)^{k-1} d(k) \right).$$

  **if** $g_h == h$ **then**

    break;

  **end if**

  $h \leftarrow g_h$

**end while**

**return** $h$, $r_h$

We now establish the following lemmas and propositions. We use them to prove the main theorem in this section, i.e., the proof of correctness and finite convergence of Algorithm 1.

Let us denote the value of variable $h$ at the beginning of $k^{\text{th}}$ iteration of Algorithm 1 by $h^{(k)}$ with $h^{(0)} = 0$. Then we prove the following

**Lemma 3.4.1.** $h^{(k)} \geq g^*$ *for* $k \geq 1$.

*Proof.* We have $h^{(k)} \overset{(1)}{=} g(h^{(k-1)}) \overset{(2)}{\geq} g^*$.
Where $(1)$ follows directly from successive iterations of the algorithm and $(2)$ follows from the fact that $g^*$ is the infimum of $g(\cdot)$. ■

Now we derive an equation characterizing the threshold policy in general. We recall that, for all $j \in \mathbb{Z}_+$, $\Delta(j) = d(j+1) - d(j)$. It follows that, for $r \geq 1$,

$$d(r) = \sum_{j=0}^{r-1} \Delta(j) + d(0) \tag{3.64}$$

Define, for $h \geq 0$,

$$r(h) = \min\{r \in \mathbb{Z}_+ : p(\lambda + h) \leq \Delta(r)\} \tag{3.65}$$

The cost to go for the threshold policy with threshold $r(h)$, starting from the state $(0, \mathsf{c})$, which we denote by $g(h)$, can be written as (suppressing the argument $(h)$ in $g(h)$ and $r(h)$, for convenience of writing)

$$g = \sum_{k=1}^{r} (1-p)^{k-1} p d(k) + (1-p)^r (\lambda + d(r) + g)$$

Denoting, for $k \geq 1$, $p_k = p(1-p)^{k-1}$, and using Eqn. (3.64), we can rewrite the previous expression as

$$\begin{aligned} g &= \sum_{k=1}^{r} p_k \left( \sum_{j=0}^{k-1} \Delta(j) + d(0) \right) + (1-p)^r (\lambda + d(r) + g) \\ &= \sum_{k=1}^{r} p_k \left( \sum_{j=0}^{k-1} \Delta(j) \right) + (1 - (1-p)^r) d(0) + (1-p)^r (\lambda + d(r) + g) \end{aligned}$$

Rearranging the order of the sums in the first term, and using Eqn. (3.64) to substitute for $d(r) - d(0)$, we obtain

$$g = \sum_{j=0}^{r-1} \left( \sum_{k=j+1}^{r} p_k + (1-p)^r \right) \Delta(j) + d(0) + (1-p)^r (\lambda + g)$$

Finally, recognising that the sum of the probabilities multiplying $\Delta(j)$ in the first expression is just $(1-p)^j$, and reintroducing the argument $(h)$, we obtain

$$g(h) = \sum_{j=0}^{r(h)-1} (1-p)^j \Delta(j) + d(0) + (1-p)^{r(h)} (\lambda + g(h)) \tag{3.66}$$

We can write this relation in an alternative way by rearranging as follows:

$$
\begin{aligned}
g(h) &= \sum_{j=0}^{r(h)-1}(1-p)^j\Delta(j)+d(0)+(1-p)^{r(h)}(\lambda+g(h)) \\
&= \sum_{j=0}^{r(h)-1}(1-p)^j\Delta(j)+d(0)+\left(1-\sum_{j=0}^{r(h)-1}(1-p)^j p\right)(\lambda+g(h)) \\
&= \sum_{j=0}^{r(h)-1}(1-p)^j(\Delta(j)-p(\lambda+g(h)))+d(0)+\lambda+g(h)
\end{aligned}
$$

which implies that

$$
0 = \sum_{j=0}^{r(h)-1}(1-p)^j\left(\Delta(j)-p(\lambda+g(h))\right)+d(0)+\lambda \tag{3.67}
$$

Using the above equation, we establish the following Lemma.

**Lemma 3.4.2.** *If $h > g^*$ then $h > g(h)$.*

*Proof.* First, consider $h > g^*$ and such that $r(h) = r^*$. But then $g(h) = g^*$, and we conclude that $h > g(h)$.

Next, consider $h > g^*$ and such that $r(h) \geq r^*+1$. From Eqn. (3.65), we conclude that

$$
p(\lambda+h) > \Delta(r(h)-1) > \Delta(r(h)-2) > \cdots \tag{3.68}
$$

$$
p(\lambda+h) \leq \Delta(r(h)) < \Delta(r(h)+1) < \cdots \tag{3.69}
$$

Applying Eqn. (3.67) to $h = g^*$, we obtain

$$
0 = \sum_{j=0}^{r^*-1}(1-p)^j\left(\Delta(j)-p(\lambda+g^*)\right)+d(0)+\lambda
$$

from which we derive

$$
\sum_{j=0}^{r^*-1}(1-p)^j\Delta(j) = \sum_{j=0}^{r^*-1}(1-p)^j p(\lambda+g^*))-(d(0)+\lambda)
$$

Then applying Eqn. (3.67) to the given $h$, we obtain

$$
\begin{aligned}
0 &= \sum_{j=0}^{r(h)-1}(1-p)^j\left(\Delta(j)-p(\lambda+g(h))\right)+d(0)+\lambda \\
&= \sum_{j=0}^{r^*-1}(1-p)^j\left(\Delta(j)-p(\lambda+g(h))\right)+\sum_{j=r^*}^{r(h)-1}(1-p)^j\left(\Delta(j)-p(\lambda+g(h))\right)+d(0)+\lambda
\end{aligned}
$$

Substituting for $\sum_{j=0}^{r^*-1}(1-p)^j\Delta(j)$ from the previous expression

$$\begin{aligned}
0 &= \sum_{j=0}^{r^*-1}(1-p)^j p(g^*-g(h)) + \sum_{j=r^*}^{r(h)-1}(1-p)^j\left(\Delta(j)-p(\lambda+g(h))\right) \\
&\leq \sum_{j=r^*}^{r(h)-1}(1-p)^j\left(\Delta(j)-p(\lambda+g(h))\right) \\
&< \sum_{j=r^*}^{r(h)-1}(1-p)^j(p(\lambda+h)-p(\lambda+g(h))) \\
&= p(h-g(h))\sum_{j=r^*}^{r(h)-1}(1-p)^j
\end{aligned}$$

where the first inequality follows from $g^* \leq g(h)$ for all $h$, and the second from Eqn. (3.68). Since $0 < p < 1$, we conclude that

$$h > g(h)$$

■

**Corollary 3.4.3.** *$g(h)$ has a unique fixed point.*

*Proof.* We know that $g(h)$ has a fixed point at $h = g^*$. Now consider the following cases

**Case 1:** $h > g^*$

In this case, from lemma 3.4.2 we know that $h > g(h)$. Hence $g(\cdot)$ can not have a fixed point which is strictly greater that $g^*$.

**Case 2:** $h < g^*$

In this case we have $h < g^* \stackrel{(1)}{\leq} g(h)$. Where $(1)$ follows from the fact that $g^* = \inf g(\cdot)$. Hence in this case we have $h < g(h)$. Thus $g(\cdot)$ can not have a fixed point which is strictly less than $g^*$.

Combining the above cases the proof follows.

■

**Lemma 3.4.4.** *The sequence $\{h^{(k)}\}_{k\geq 1}$ is non-increasing i.e. $h^{(k+1)} \leq h^{(k)}$, with the equality sign holding iff $h^{(k)} = g^*$.*

*Proof.* We have for all $k \geq 1$, $h^{(k+1)} \stackrel{(1)}{=} g(h^{(k)}) \stackrel{(2)}{\leq} h^{(k)}$

Here (1) follows from successive iterations of the fixed point iterations and (2) follows from lemma 3.4.1 and 3.4.2. Clearly the equality sign in the lemma holds iff equality in (2) holds, i.e. iff $h^{(k)}$ is a fixed point of $g(\cdot)$. Then from corollary 3.4.3 it follows that $h^{(k+1)} = h^{(k)}$ iff $h^{(k)} = g^*$.

■

**Lemma 3.4.5.** *The sequence* $\{r(h^{(k)})\}_{k\geq 1}$ *is non-increasing i.e.* $r(h^{(k+1)}) \leq r(h^{(k)})$*, with the equality sign holding iff* $r(h^{(k+1)}) = r^*$.

*Proof.* Recall that

$$r(h) = \min\{r \in \mathbb{Z}_+ : p(\lambda + h) \leq \Delta(r)\}$$

Now take $h_2 > h_1$. We have

$$\begin{aligned} r(h_2) &= \min\{r \in \mathbb{Z}_+ : p(\lambda + h_2) \leq \Delta(r)\} && (3.70) \\ &\geq \min\{r \in \mathbb{Z}_+ : p(\lambda + h_1) \leq \Delta(r)\} && (3.71) \\ &= r(h_1) && (3.72) \end{aligned}$$

Hence the function $r(h)$ is non-decreasing in $h$. Thus from Lemma 3.4.4 it follows that the sequence $\{r(h^{(k)})\}_{k\geq 1}$ is non-increasing.

Now note that the function $g(h)$ depends on $h$ only through $r(h)$. Hence if $r(h^{(k)}) = r(h^{(k+1)})$, we must have $g(h^{(k+1)}) = g(h^{(k)}) = h^{(k+1)}$. Hence from Lemma 3.4.3, it follows that $h^{(k+1)} = g^*$, and hence $r(h^{(k+1)}) = r^*$. ■

Now we state and prove the main theorem in this section.

**Theorem 3.4.6.** *Algorithm 1 returns with* $g^*$ *and* $r^*$ *in finite time.*

*Proof.* From Lemma 3.4.5, it follows that starting with a finite integer $r(h^{(1)})$, the sequence $\{r(h^{(k)})\}$ decreases strictly at each iteration until it reaches the optimum $r^*$. At this point we have $h^{(k)} = g(h^{(k)}) = g^*$ and the algorithm terminates. ■

## 3.5 Solving the Constrained MDP

Once we get the optimal threshold $r^*_\lambda$, we can calculate the expected number of relays used, $\mathbb{E}_{\pi^*_\lambda} N$, following a similar argument as in (3.54), we get

$$\mathbb{E}_{\pi^*_\lambda} N = \frac{(p+v)^{r^*_\lambda}}{1-(1-(p+v))^{r^*_\lambda}}. \tag{3.73}$$

We now invoke Lemma 2.2.1 to provide a solution for the constrained placement problem (2.2) on the random lattice path, with initial state $(0,0)$. We begin by making following observations about $\mathbb{E}_{\pi^*_\lambda} N$.

1) $\mathbb{E}_{\pi^*_\lambda} N$ decreases with $\lambda$; this is as expected, since as each relay becomes "costlier" fewer relays are used on the average.

2) Even when $\lambda = 0$, $\mathbb{E}_{\pi^*_\lambda} N$ is finite. This is because $d(0) > 0$, i.e., there is a positive cost for a 0 length link. Define the value of $\mathbb{E}_{\pi^*_\lambda} N$ with $\lambda = 0$ to be $\rho_{\max}$.

3) $\mathbb{E}_{\pi^*_\lambda} N$ vs. $\lambda$ is a piecewise constant function. This occurs because the relay placement positions are discrete. For a range of values of $\lambda$ the same threshold is optimal. This structure is also evident from the results based on the optimal stopping formulation and the OSLA rule in Section 3.2.2. It follows that for a value of $\lambda$ at which there is a step in the plot, there are two optimal deterministic policies, $\underline{\pi}$ and $\overline{\pi}$, for the relaxed problem. Let $\underline{\rho} = \mathbb{E}_{\underline{\pi}} N$ and $\overline{\rho} = \mathbb{E}_{\overline{\pi}} N$.

We have the following structure of the optimal policy for the constrained problem:

**Theorem 3.5.1.** *(i) For $\rho_{avg} \geq \rho_{max}$ the optimal placement threshold is $r^*_0$.*
*(ii) For $\rho_{avg} < \rho_{max}$, if there is a $\lambda$ such that (a) $\mathbb{E}_{\pi^*_\lambda} N = \rho_{avg}$ then the optimal policy is $\pi^*_\lambda$, or (b) $\underline{\rho} < \rho_{avg} < \overline{\rho}$ then the optimal policy is obtained by mixing $\underline{\pi}$ and $\overline{\pi}$.*

*Proof.* (i) is straight forward. For proof of (ii)-(a), see Lemma 2.2.1. Define $0 < \alpha < 1$ such that $(1-\alpha)\underline{\rho} + \alpha\bar{\rho} = \rho_{avg}$. We obtain a mixing policy $\pi_m$ by choosing $\underline{\pi}$ w.p. $1-\alpha$ and $\bar{\pi}$ w.p. $\alpha$ at the beginning of the deployment. For any policy $\pi$ we have

$$
\begin{aligned}
&\mathbb{E}_{\pi_m} C + \lambda \mathbb{E}_{\pi_m} N \\
&= (1-\alpha)(\mathbb{E}_{\underline{\pi}} C + \lambda \underline{\rho}) + \alpha(\mathbb{E}_{\bar{\pi}} C + \lambda \bar{\rho}) \\
&\leq (1-\alpha)(\mathbb{E}_{\pi} C + \lambda \mathbb{E}_{\pi} N) + \alpha(\mathbb{E}_{\pi} C + \lambda \mathbb{E}_{\pi} N) \\
&= \mathbb{E}_{\pi} C + \lambda \mathbb{E}_{\pi} N
\end{aligned}
\tag{3.74}
$$

The inequality is because $\underline{\pi}$ and $\overline{\pi}$ are both optimal for the problem (2.1) with relay price $\lambda$. Thus, we have shown that $\pi_m$ is also optimal for the relaxed problem. Using this along with $\mathbb{E}_{\pi_m} N = \rho_{avg}$ in Lemma 2.2.1 we conclude the proof. ∎

## 3.6 Summary

In this chapter, we studied the problem of impromptu relay deployment along a corridor with LOS propagation. In section 3.1, we formulated the problem as an Infinite Horizon Total Cost MDP. In section 3.3 we solved the relaxed problem and showed the equivalence of the problem with the single corridor problem solved by Prasenjit et al. in [13]. In section 3.3.2, we formulated the problem as an optimal stopping problem and proved the optimality of the One-Step-Look-Ahead policy in Theorem 3.3.5. We developed a simple expression for calculating the threshold in section 3.4, based on a renewal argument. Finally, in section 3.5 we solved the original constrained MDP from the solution of the relaxed problem.

# Chapter 4

# The Non-Line-Of-Sight Case

In this section we consider relay placement along a random lattice path, where a usable radio link can exist between any pair of points on the path, even though they may not be in line-of-sight along the path. The traffic model, network operation model, and the per-hop cost model described in Section 2.1 continue to hold. We formulate the problem as a total cost infinite horizon MDP and show that the optimal placement set corresponds to a two-dimensional boundary upon crossing which a relay must be placed. We characterize this boundary via a formulation as an optimal stopping problem, and study its sensitivity to some parameters of the problem.

## 4.1 State Space, Actions, and Transition Structure

We formulate the problem as a sequential decision process starting at the entrance to the building (also the entrance of the corridor), i.e., at the point $(x_0, 0)$, where we recall that $x_0 \geq 1$. The decision points are indexed by $k \in \{0, 1, 2, \cdots\}$, with $k = 0$ corresponding to the decision to be made at the point $(x_0, 0)$. We also refer to the $k^{\text{th}}$ decision point as the location $k$. For $k \geq 0$, let $X_k = (m_k, n_k)$ denote the coordinates of placement operative with respect to the previous relay (or the Base-Station) where $m_k$ is the number of steps East and $n_k$ is the number of step North.
Let, for $k \geq 0$, $Z_k \in \{\mathsf{e}, \mathsf{c}\}$, where $Z_k = \mathsf{e}$ means at Step $k$, the corridor has ended and $Z_k = \mathsf{c}$ denotes that the corridor will continue at least for one more step. Since the corridor extends at least one step beyond its entrance, we see that $Z_0 = \mathsf{c}$. The state $\{\phi\}$ denotes the cost-free termination state, i.e., the state after the end of the corridor has been discovered. The state-space is given by

$$\mathscr{S} = \{(m,n,\mathsf{c}), (m,n,\mathsf{e}\} \bigcup \{\phi\} \quad (m,n) \in \mathbb{Z}_+^2 \tag{4.1}$$

Finally by the sequence $U_k \in \mathbb{A}_k$, we denote the sequence of actions. The action $U_k = 1$ denotes the action of placing a relay node at the $k^{\text{th}}$ step and $U_k = 0$ denotes the action of not placing a relay node at the $k^{\text{th}}$ step. The permissible action sets are as follows

- $\mathbf{A}_{(x,z)} = \{0,1\}$ for $z = \mathsf{c}$
- $\mathbf{A}_{(x,z)} = \{1\}$ for $z = \mathsf{e}$

To simplify the notation, hereafter drop $Z_k$ from the state notation if $Z_k = \mathsf{c}$.
Given that $(X_k, Z_k) = (m, n, z)$, and if the action $U_k$ is taken, we now write down the state transition probabilities The state-transitions are shown as follows

- If $u_k$ is 0,

$$(m,n) \longrightarrow (m+1,n) \quad \text{w.p.} \quad (1-p)q \tag{4.2}$$
$$(m,n) \longrightarrow (m+1,n,\mathsf{e}) \quad \text{w.p.} \quad pq \tag{4.3}$$
$$(m,n) \longrightarrow (m,n+1) \quad \text{w.p.} \quad (1-p)(1-q) \tag{4.4}$$
$$(m,n) \longrightarrow (m,n+1,\mathsf{e}) \quad \text{w.p.} \quad p(1-q) \tag{4.5}$$

- If $u_k$ is 1,

$$(m,n) \longrightarrow (1,0) \quad \text{w.p.} \quad (1-p)q \tag{4.6}$$
$$(m,n) \longrightarrow (1,0,\mathsf{e}) \quad \text{w.p.} \quad pq \tag{4.7}$$
$$(m,n) \longrightarrow (0,1) \quad \text{w.p.} \quad (1-p)(1-q) \tag{4.8}$$
$$(m,n) \longrightarrow (0,1,\mathsf{e}) \quad \text{w.p.} \quad p(1-q) \tag{4.9}$$

- If $Z_k = \mathsf{e}$ the only allowable action is $u_k = 1$ and we enter into the state $\{\phi\}$ w.p. 1

$$(m,n,\mathsf{e}) \longrightarrow \{\phi\} \quad \text{w.p.} \quad 1 \tag{4.10}$$

- If the current state is $\phi$ we stay in the same cost-free termination state irrespective of the control $u_k$

$$\{\phi\} \longrightarrow \{\phi\} \quad \text{w.p.} \quad 1 \tag{4.11}$$

If $Z_k = \mathsf{e}$ then the only allowable action is $u = 1$ and we enter into the state $\{\phi\}$. If the current state is $\phi$ we stay in the same cost-free termination state irrespective of the control $u$. Then one step cost when the state is $s$ is given by:

$$c(s,u) = \begin{cases} d(m,n) & \text{if } s = (m,n,\mathsf{e}) \\ \lambda + d(m,n) & \text{if } u = 1 \text{ and } s = (m,n,\mathsf{c}) \\ 0 & \text{if } u = 0 \text{ or } s = \phi. \end{cases}$$

## 4.2 Solving the Relaxed Problem

### 4.2.1 Optimal Placement

For simplicity we will write the state $(m,n,c)$ as simply $(m,n)$. Let $J_\lambda(m,n)$ denote the optimal cost-to-go when the current state is $(m,n)$. When at some step the state is $(m,n)$ the deployment person has to decide whether to place or not place at relay at the current step. By [5, Prop. 1.1, Page 137] $J_\lambda(\cdot,\cdot)$ is a solution to Bellman's equation,

$$J_\lambda(m,n) = \min\{c_p(m,n), c_{np}(m,n)\} \tag{4.12}$$

where $c_p(m,n)$ and $c_{np}(m,n)$ denotes the expected cost incurred when the decision is to place and not place a relay, respectively. $c_p(m,n)$ is given by

$$c_p(m,n) = \lambda + d(m,n) + (1-p)(1-q)J_\lambda(0,1) + (1-p)qJ_\lambda(1,0) + pd(1). \tag{4.13}$$

The above equation for $c_p(m,n)$ may be understood as follows. If we place a relay at the state $(m,n)$ we incur a relay cost of $\lambda$ and a delay of $d(m,n)$. This accounts for the first two terms in $c_p(m,n)$. After placing the relay there are four possibilities, viz

- The path continues for one more step in the eastward direction and does not end, thus reaching the state $(1,0)$. This event has a probabibility of occurance $(1-p)q$ and cost-to-go $J_\lambda(1,0)$. This accounts for the third term.
- The path continues for one more step in the eastward direction and ends there. This event has a probability of occurance $pq$ and incurs a delay cost of $d(1)$. (We do not count the cost of the relay at the end of the path.) This accounts for the fourth term.
- The path continues for one more step in the northward direction and does not end, thus reaching the state $(0,1)$. This event has a probabibility of occurance $(1-p)(1-q)$ and cost-to-go $J_\lambda(1,0)$. This accounts for the fifth term.
- The path continues for one more step in the northward direction and ends there. This event has a probability of occurance $p(1-q)$ and incurs a delay cost of $d(1)$. This accounts for the last term in the expression of $c_p(m,n)$.

Similarly,

$$c_{np}(m,n) = (1-p)qJ_\lambda(m+1,n) + pqd(m+1,n) + (1-p)(1-q)J_\lambda(m,n+1) + p(1-q)d(m,n+1) \quad (4.14)$$

The terms in the above expression may also be explained similarly as before. If we denote the optimal placement set by $\mathscr{P}_\lambda$, then

$$\mathscr{P}_\lambda = \{(m,n) : c_p(m,n) \leq c_{np}(m,n)\} \quad (4.15)$$

After some algebraic manipulations, we finally arrive at the following expression for $\mathscr{P}_\lambda$

$$\begin{aligned}\mathscr{P}_\lambda = \{(m,n) : (1-p)(qH(m+1,n) + (1-q)H(m,n+1)) + q\Delta_1(m,n) + (1-q)\Delta_2(m,n) \\ \geq \lambda + (1-p)qJ_\lambda(1,0) + (1-p)(1-q)J_\lambda(0,1) + pd(1)\}\end{aligned} \quad (4.16)$$

Where $H(m,n) := J_\lambda(m,n) - d(m,n)$.

Now we prove the following Lemma.

**Lemma 4.2.1.** *$H(m,n) \equiv J_\lambda(m,n) - d(m,n)$ is non decreasing in both $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+$.*

*Proof.* Consider a sequential relay placement problem where we have $K$ steps to go. Hence, the corridor length is the min of $K$ and the value sampled from Geom($p$). This can be formulated as a finite horizon MDP with horizon length $K$. For any given $(m,n)$, we may write $J_K(m,n)$, $K \geq 2$ recursively as

$$\begin{gathered}J_K(m,n) = \min\{c_p(m,n), c_{np}(m,n)\} \\ \min\{\lambda + d(m,n) + (1-p)qJ_{K-1}(1,0) + pqd(1) + (1-p)(1-q)J_{K-1}(0,1) + p(1-q)d(1), \\ (1-p)qJ_{K-1}(m+1,n) + pqd(m+1,n) + (1-p)(1-q)J_{K-1}(m,n+1) + p(1-q)d(m,n+1)\}\end{gathered}$$

For $K = 1$, since a sensor must be placed at the next step, we have

$$J_1(m,n) = \min\{\lambda + d(m,n) + d(1), qd(m+1,n) + (1-q)d(m,n+1)\}$$

so,

$$\begin{aligned}H_1(m,n) &:= J_1(m,n) - d(m,n) \\ &= \min\{\lambda + d(1), q(d(m+1,n) - d(m,n)) + (1-q)(d(m,n+1) - d(m,n))\} \\ &= \min\{\lambda + d(1), q\Delta_1(m,n) + (1-q)\Delta_2(m,n)\}\end{aligned}$$

From the theorem (2.2.2), it follows that $H_1(m,n)$ is non decreasing in both $m$ and $n$. Now we make the induction

hypothesis and assume that $H_{K-1}(m,n)$ is non decreasing in $m$ and $n$. We show that $H_K(m,n)$ is also non decreasing in $m$ and $n$.
We have,

$$H_K(m,n) = J_K(m,n) - d(m,n)$$
$$\min\{\lambda + (1-p)qJ_{K-1}(1,0) + pqd(1) + (1-p)(1-q)J_{K-1}(0,1) + p(1-q)d(1),$$
$$(1-p)(qH_{K-1}(m+1,n) + (1-q)H_{K-1}(m,n+1)) + q(d(m+1,n) - d(m,n)) + (1-q)(d(m,n+1) - d(m,n))\}$$

$$= \min\{\lambda + (1-p)qJ_{K-1}(1,0) + pqd(1) + (1-p)(1-q)J_{K-1}(0,1) + p(1-q)d(1),$$
$$(1-p)(qH_{K-1}(m+1,n) + (1-q)H_{K-1}(m,n+1)) + q\Delta_1(m,n) + (1-q)\Delta_2(m,n)\}$$

By the induction hypothesis and theorem (2.2.2), it follows that $H_K(m,n)$ is non decreasing in both $m$ and $n$. Using Proposition 1.6 of ([5]), the proof is complete by taking the limit as $K \to \infty$. ■

The above result yields the following theorem which characterizes the optimal placement set $\mathscr{P}_\lambda$

**Theorem 4.2.2.** *The optimal placement policy is a threshold policy, i.e., there exist mappings $m^* : \mathbb{Z}_+ \to \mathbb{Z}_+$ and $n^* : \mathbb{Z}_+ \to \mathbb{Z}_+$ which defines the optimal placement set $\mathscr{P}_\lambda$ as follows,*

$$\mathscr{P}_\lambda = \bigcup_{n \in \mathbb{Z}_+} \{(m,n) | m \geq m^*(n)\} \tag{4.17}$$
$$= \bigcup_{m \in \mathbb{Z}_+} \{(m,n) | n \geq n^*(m)\} \tag{4.18}$$

*Here $\mathbb{Z}_+$ denotes the set of non negative integers.*

*Proof.* Referring to (4.16), utilizing Lemma (4.2.1) and the Theorem (2.2.2), it follows that for a fixed $n \in \mathbb{Z}_+$, the LHS of the inequality (4.16), describing the placement set $\mathscr{P}_\lambda$ is an increasing function of $m$, while the RHS is a *finite* constant. Also, because of the assumed properties of the function $d()$, $\Delta_1(m,n) \to \infty$ as $m \to \infty$, for any fixed $n$. Hence it follows that there exists an $m^*(n) \in \mathbb{Z}_+$ such that $(m,n) \in \mathscr{P}_\lambda \quad \forall m \geq m^*(n)$. Hence we may write

$$P_\lambda = \bigcup_{n \in \mathbb{Z}_+} \{(m,n) | m \geq m^*(n)\} \tag{4.19}$$

The second characterization follows by similar arguments. ■

We have the following immediate corollary from the above theorem.

**Corollary 4.2.3.** *$m^*(n)$ is non-increasing in $n$ and $n^*(m)$ is non-increasing in $m$. ($(m,n) \in \mathbb{Z}_+^2$)*

## 4.3 Optimal Stopping Formulation

The transparent-wall problem also may be formualted as an optimal stopping problem exactly like that of the opaque-wall problem as is done in section 3.4. In Section 4.0.1 and 4.0.2, we observed that the relaxed problem in the context of Radio Transparent Wall, shown in Equation (2.1) is a total cost Infinite Horozon Markov decision process. The problem terminates, with probability one, with the corridor ending and a sensor being placed at this point. The time homogeneous structure of the problem permits us to conclude that each point at which a relay is placed (the corridor not having ended) is a renewal point, from where the cost to go is $J(0,0)$. Now consider the decision process as the operative walks after having placed a relay. If after $x$ steps the path has not ended, there are two actions possible: to place a relay, which can be viewed as a "stop" action, upon taking which a "termination" cost $J(0,0)$ is incurred, or to not place a relay, which can be viewed as a "continue" action. If a relay is not placed then another step is taken after which the corridor may end, with probability $p$, and a sensor is placed; with probability $(1-p)$ the corridor does not end and another decision to stop or to continue has to be taken. We thus see that between relay placements (or between the base-station and the first relay placement) we have an optimal stopping problem. Consider now the one-step-look-ahead policy discussed below.

In the One-Step-Look-Ahead policy, we compare the costs of placing a relay at the current step with that of continuing without placing in the current step and placing in the next step instead. Let us call the resulting placement set $\bar{\mathscr{P}}_\lambda$ defined as follows,

$$\begin{aligned}\bar{\mathscr{P}}_\lambda &= \{(m,n)\in\mathbb{Z}_+^2 | \lambda + d(m,n) + J^*(0,0) \le p(qd(m+1,n)+(1-q)d(m,n+1)) + \\ &\qquad (1-p)(qd(m+1,n)+(1-q)d(m,n+1)+\lambda+J^*(0,0))\} \\ &= \{(m,n)\in\mathbb{Z}_+^2 | p(\lambda + J_\lambda(0,0)) \le qd(m+1,n)+(1-q)d(m,n+1)-d(m,n)\} \\ &= \{(m,n)\in\mathbb{Z}_+^2 | p(\lambda + J_\lambda(0,0)) \le q\Delta_1(m,n)+(1-q)\Delta_2(m,n)\}\end{aligned} \tag{4.20}$$

It can be seen from the theorem (2.2.2), the placement set defined by (4.20) also implements a threshold policy, i.e. the set (4.20) may be written as follows

$$\bar{\mathscr{P}}_\lambda = \bigcup_{n\in\mathbb{Z}_+} \{(m,n) | m \ge \bar{m}(n)\} \tag{4.21}$$

$$= \bigcup_{m\in\mathbb{Z}_+} \{(m,n) | n \ge \bar{n}(m)\} \tag{4.22}$$

For some mappings $\bar{m}:\mathbb{Z}_+\to\mathbb{Z}_+$ and $\bar{n}:\mathbb{Z}_+\to\mathbb{Z}_+$. We have the following Theorem. In the following we show that policy obtained from solving the detailed Bellman Equation and from the OSLA rule are equivalent. We establish this result by the series of lemmas as follows. Recall that $\mathscr{P}_\lambda$ is the placement set obtained from solving the detailed Bellman Equation and $\bar{\mathscr{P}}_\lambda$ is the placement set obtained from the OSLA policy.

**Lemma 4.3.1.** $\mathscr{P}_\lambda \subset \bar{\mathscr{P}}_\lambda$

*Proof.* Suppose that $(m,n) \in \mathbb{Z}_+^2$ and $(m,n) \in \mathscr{P}_\lambda$.

Then from Eqn. (4.17) $(m+1,n) \in \mathscr{P}_\lambda$ and from Eqn. (4.18),$(m,n+1) \in \mathscr{P}_\lambda$.

Since $(m,n) \in \mathscr{P}_\lambda$, we have from Eqns. (4.13), (4.14) and (4.15), that

$$\begin{aligned} &\lambda + d(m,n) + (1-p)qJ_\lambda(1,0) + pqd(1) + (1-p)(1-q)J_\lambda(0,1) + p(1-q)d(1) \leq \\ &(1-p)qJ_\lambda(m+1,n) + pqd(m+1,n) + (1-p)(1-q)J_\lambda(m,n+1) + p(1-q)d(m,n+1) \end{aligned} \tag{4.23}$$

Also we may similarly argue as in Proposition (3.2.3) to prove that at the state $(0,0)$, it is optimal not to place. Hence,

$$J_\lambda(0,0) = (1-p)qJ_\lambda(1,0) + pqd(1) + (1-p)(1-q)J_\lambda(0,1) + p(1-q)d(1) \tag{4.24}$$

Since $(m+1,n) \in \mathscr{P}_\lambda$ and $(m,n+1) \in \mathscr{P}_\lambda$, we have (utilizing Eqn. (4.13) and (4.24)),

$$J_\lambda(m+1,n) = \lambda + d(m+1,n) + J_\lambda(0,0) \tag{4.25}$$

$$J_\lambda(m,n+1) = \lambda + d(m,n+1) + J_\lambda(0,0) \tag{4.26}$$

Now we combine (4.23), (4.25) and (4.26), to obtain

$$p(\lambda + J_\lambda(0,0)) \leq q\Delta_1(m,n) + (1-q)\Delta_2(m,n) \tag{4.27}$$

This proves that

$$\begin{aligned} &(m,n) \in \bar{\mathscr{P}}_\lambda \\ \text{And hence,} \quad &\mathscr{P}_\lambda \subset \bar{\mathscr{P}}_\lambda \end{aligned} \tag{4.28}$$

■

Taking into account the threshold structure of the sets (viz Eqns (4.17), (4.18), (4.21), (4.22)) the above lemma implies that

$$n^*(m) \geq \bar{n}(m) \quad \forall m \in \mathbb{Z}_+ \tag{4.29}$$

$$m^*(n) \geq \bar{m}(n) \quad \forall n \in \mathbb{Z}_+ \tag{4.30}$$

We now prove the following technical lemma which characterizes the set $\mathscr{P}_\lambda$.

**Lemma 4.3.2.** *Let $N \equiv (m,n) \in \mathbb{Z}_+^2$ and $(m,n) \in \bar{\mathscr{P}}_\lambda$ with $(m,n+1) \in \mathscr{P}_\lambda$ and $(m+1,n) \in \mathscr{P}_\lambda$, then $N \in \mathscr{P}_\lambda$*

In words, if a lattice point $N \equiv (m,n)$ lies in the set $\bar{\mathscr{P}}_\lambda$ with its northward and eastward neighbour belonging to the set $\mathscr{P}_\lambda$, then the point $N$ also lies in the set $\mathscr{P}_\lambda$.

*Proof.* If possible, suppose $(m,n) \notin \mathscr{P}_\lambda$. Then we have the following inclusion relations valid

$$(m,n) \in \bar{\mathscr{P}}_\lambda \tag{4.31}$$

$$(m,n) \notin \mathscr{P}_\lambda \tag{4.32}$$

$$(m,n+1) \in \mathscr{P}_\lambda \tag{4.33}$$

$$(m+1,n) \in \mathscr{P}_\lambda \tag{4.34}$$

Since $(m,n) \in \bar{\mathscr{P}}_\lambda$, we have from Eqns. (4.20),

$$p(\lambda + J_\lambda(0,0)) \leq q\Delta_1(m,n) + (1-q)\Delta_2(m,n) \tag{4.35}$$

Now $(m,n+1) \in \mathscr{P}_\lambda$, and $(m+1,n) \in \mathscr{P}_\lambda$, hence we have from Eqns (4.25) and (4.26)

$$J_\lambda(m+1,n) = \lambda + d(m+1,n) + J_\lambda(0,0)$$
$$J_\lambda(m,n+1) = \lambda + d(m,n+1) + J_\lambda(0,0)$$

Now we can work backwards the steps in lemma (4.3.1) to obtain

$$\lambda + d(m,n) + (1-p)qJ_\lambda(1,0) + pqd(1) + (1-p)(1-q)J_\lambda(0,1) + p(1-q)d(1) \leq$$
$$(1-p)qJ_\lambda(m+1,n) + pqd(m+1,n) + (1-p)(1-q)J_\lambda(m,n+1) + p(1-q)d(m,n+1)$$

But it was assumed that $(m,n) \notin \mathscr{P}_\lambda$. Hence we may write from Eqns. (4.13) and (4.14) that

$$\lambda + d(m,n) + (1-p)qJ_\lambda(1,0) + pqd(1) + (1-p)(1-q)J_\lambda(0,1) + p(1-q)d(1) >$$
$$(1-p)qJ_\lambda(m+1,n) + pqd(m+1,n) + (1-p)(1-q)J_\lambda(m,n+1) + p(1-q)d(m,n+1) \tag{4.36}$$

Eqns (4.36) and (4.36) contradicts. Hence what we assumed was incorrect and indeed $(m,n) \in \mathscr{P}_\lambda$ ■

Now we use the above lemma to prove the following

**Lemma 4.3.3.** $\bar{\mathscr{P}}_\lambda \subset \mathscr{P}_\lambda$

*Proof.* Keeping in mind the threshold structure of the sets ((4.17), (4.18), (4.21), (4.22)), we only need to show that the inequalities in Eqns (4.29) and (4.30) are in fact equalities.
We fix an $m \in \mathbb{Z}_+$ and show the first inequality (4.29) is a equality, i.e. $n^*(m) = \bar{n}(m)$. We again prove by contradiction.

If possible, let $n^*(m) > n^*(m)-1 \geq \bar{n}(m)$. Then we have the following inclusion relations valid (by the characterization (4.26))

$$(m, n^*(m)) \in \mathscr{P}_\lambda \tag{4.37}$$

$$(m, n^*(m)-1) \in \bar{\mathscr{P}}_\lambda \tag{4.38}$$

$$(m, n^*(m)-1) \notin \mathscr{P}_\lambda \tag{4.39}$$

Let us index the collection of lattice-points $(m+i, n^*(m)-1)$ by $N_i \in \bar{\mathscr{P}}_\lambda, i \in \mathbb{Z}_+$. Then by the characterization (4.25), we conclude that there exists a finite $k \in \mathbb{Z}_+$, s.t. $N_k \in \mathscr{P}_\lambda$. Hence we have the following inclusion relations valid

$$(m+k, n^*(m)-1) \in \mathscr{P}_\lambda \tag{4.40}$$

$$(m+k-1, n^*(m)) \in \mathscr{P}_\lambda \tag{4.41}$$

The last inclusion relation holds because of characterization (4.25) and noting that $(m^*(n^*(m)) = m$.
Now we see that for the point $N_{k-1}$, the conditions of lemma (4.3.2) are satisfied. Hence $N_{k-1} \in \mathscr{P}_\lambda$. This in turn makes the point $N_{k-2}$ subject to the condition of lemma (4.3.2) (Remembering that $(m+k-2, n^*(m)) \in \mathscr{P}_\lambda$) and hence $N_{k-2} \in \mathscr{P}_\lambda$. Working this way (right to left) we ultimately conclude that $N_0 \equiv (m, n^*(m)-1) \in \mathscr{P}_\lambda$. This contradicts with equation (4.39) and proves the result. ■

Combining lemmas (4.3.1) and (4.3.3) we have the following theorem

**Theorem 4.3.4.** $\mathscr{P}_\lambda = \bar{\mathscr{P}}_\lambda$

We now prove a general monotonicity property of the cost-to-go functions $J_\lambda(0,0)$.

**Proposition 4.3.5.** $J_{\lambda_2}(0,0) \geq J_{\lambda_1}(0,0)$ *for* $\lambda_2 \geq \lambda_1$

*Proof.* Assume that the policies $\pi_2$ and $\pi_1$ are an optimal policy for relay-cost $\lambda_2$ and $\lambda_1$. Then we may write

$$\begin{aligned} J_{\lambda_2}(0,0) &= \mathbb{E}_{\pi_2} C + \lambda_2 \mathbb{E}_{\pi_2} N && (4.42) \\ &\geq \mathbb{E}_{\pi_2} C + \lambda_1 \mathbb{E}_{\pi_2} N && (4.43) \\ &\geq \mathbb{E}_{\pi_1} C + \lambda_1 \mathbb{E}_{\pi_1} N && (4.44) \\ &= J_{\lambda_1}(0,0) && (4.45) \end{aligned}$$

Where inequality 4.43 follows from the fact $\lambda_2 \geq \lambda_1$ and inequality 4.44 follows from the fact that policy $\pi_1$ is optimal for relay-price $\lambda_1$. ■

The following lemma proves the *monotonicity* property of the sets $\mathscr{P}_\lambda$ w.r.t $\lambda$.

**Lemma 4.3.6.** *For all $\lambda_2 > \lambda_1 > 0$ we have*

$$\mathscr{P}_{\lambda_2} \subset \mathscr{P}_{\lambda_1} \tag{4.46}$$

*Proof.* We make use of the propostion (4.3.5), which states that $J_{\lambda_2}(0,0) \geq J_{\lambda_1}(0,0)$. Again since $\lambda_2 \geq \lambda_1$, we have

$$J_{\lambda_2}(0,0) + \lambda_2 \geq J_{\lambda_1}(0,0) + \lambda_1 \tag{4.47}$$

Now let $(m,n) \in \mathbb{Z}_+^2$ s.t. $(m,n) \in \mathscr{P}_{\lambda_2}$. Hence from (4.20), we have that

$$\begin{aligned} q\Delta_1(m,n) + (1-q)\Delta_2(m,n) &\geq p(\lambda_2 + J_{\lambda_2}(0,0)) \\ &\geq p(\lambda_1 + J_{\lambda_1}(0,0)) \end{aligned}$$

Thus,

$$(m,n) \in \mathscr{P}_{\lambda_1} \tag{4.48}$$

Hence we get,

$$\mathscr{P}_{\lambda_2} \subset \mathscr{P}_{\lambda_1} \tag{4.49}$$

■

We have the following direct corollary from the above lemma.

**Corollary 4.3.7.** *For a fixed n, $m^*(n)$ is non decreasing with $\lambda$. Similar is the case for $n^*(m)$.*

The following lemma proves the symmetry of $J_{\lambda,q}(0,0)$ w.r.t. the point $q = \frac{1}{2}$.

**Lemma 4.3.8.** $J_{\lambda,q}(0,0) = J_{\lambda,1-q}(0,0)$

*Proof.* This can be shown from the geometry of the placement set. From the OSLA policy, we know that the optimal placement set is given as

$$\mathscr{P}_\lambda = \{(m,n) \in Z_+^2 | p(\lambda + J_{\lambda,q}(0,0)) \leq qd(m+1,n) + (1-q)d(m,n+1) - d(m,n)\} \tag{4.50}$$

Now we relabel the co-ordinates. This will not change the optimal placement set. So we replace $q$ by $1-q$ and $m$ by $n$ and vice-versa. Now remembering that $d(m,n) = d(n,m)$, we see that the optimal placement set is given as

$$\mathscr{P}'_\lambda = \quad \{(m,n) \in Z_+^2 | p(\lambda + J_{\lambda,1-q}(0,0)) \leq qd(m+1,n) + (1-q)d(m,n+1) - d(m,n)\} \tag{4.51}$$

But as argued before, we must have $\mathscr{P}_\lambda = \mathscr{P}'_\lambda$. Hence, comparing the above two sets, we see that we must have

$$J_{\lambda,q}(0,0) = J_{\lambda,1-q}(0,0) \tag{4.52}$$

■

**Corollary 4.3.9.** *$q = \frac{1}{2}$ is a stationary point of $J_{\lambda,q}(0,0)$*

This follows from previous theorem. We have,

$$J_{\lambda,q}(0,0) = J_{\lambda,1-q}(0,0)$$

Assuming $J_{\lambda,q}(0,0)$ is differentiable w.r.t. $q$, we differentiate both sides of the above equation w.r.t. $q$ to obtain

$$J_{\lambda,q}(0,0)' = -J_{\lambda,1-q}(0,0)'$$

Putting $q = \frac{1}{2}$, we get

$$\begin{aligned} J_{\lambda,\frac{1}{2}}(0,0)' &= -J_{\lambda,\frac{1}{2}}(0,0)' \\ or, \quad J_{\lambda,\frac{1}{2}}(0,0)' &= 0 \end{aligned} \tag{4.53}$$

Establishing the fact that $q = \frac{1}{2}$ is a stationary point of $J_{\lambda,q}(0,0)$. ■

We now observe from simulation results that $J_{\lambda,q}(0,0)$ monotonically decreases with $q \in (0, \frac{1}{2})$ and increases monotonically in $q \in (\frac{1}{2}, 1)$. Using this result we prove our next theorem

**Theorem 4.3.10.** *$m^*(0)$ decreases with $q \in [0, \frac{1}{2}]$*

*Proof.* To establish this theorem we look into the placement set in $\mathbb{R}^2$ and take the floor of the reals to get the actual boundary. Let the smooth-curve, defining the Placement set in $\mathbb{R}^2$ cuts the horizontal axis at $x^*(0)$. Putting $y = 0$ and for a certain $q$ , in the above Placement set $\mathscr{P}_\lambda$, we get an equation for $x^*(0) \equiv x^*_q(0)$ (We always have an equality for the boundary).

$$p(\lambda + J_{\lambda,q}(0,0)) = qd(x^*_q(0) + \delta, 0) + (1-q)d(x^*_q(0), \delta) - d(x^*_q(0), 0) \tag{4.54}$$

Solving the above equation, we get an explicit expression for $x^*_q(0)$. We differentiate both sides of equation [4.54], w.r.t. $q$ and substitute $d(x,y) = h(x^2 + y^2)$, where $r(.)$ is another convex increasing function whose existence has been assumed previously. We get ,

$$pJ_{\lambda,q}(0,0)' = d(x_q^*(0)+\delta,0) - d(x_q^*(0),\delta) + 2q(x_q^*(0)+\delta)x_q^*(0)'h'((x_q^*(0)+\delta)^2)+$$
$$2(1-q)x_q^*(0)x_q^*(0)'h'(x_q^*(0)^2+\delta^2) - 2x_q^*(0)x_q^*(0)'h'(x_q^*(0)^2) \geq d(x_q^*(0)+\delta,0)$$
$$d(x_q^*(0),\delta) + 2x_q^*(0)x_q^*(0)'(qh'((x_q^*(0)+\delta)^2) + (1-q)h'(x_q^*(0)^2,\delta^2) - h'(x_q^*(0)^2) \quad (4.55)$$

Now for simplicity of notation, let $x_q^*(0) \equiv x$. Then the RHS above becomes

$$h((x+\delta)^2) - h(x^2+\delta^2) + 2xx'[qh'((x+\delta)^2) + (1-q)h'(x^2+\delta^2) - h'(x^2)]$$

Since $h(.)$ is increasing in its argument, the first term $h((x+\delta)^2) - h(x^2+\delta^2) > 0$. On the other hand, we have assumed that the third derivative of $h()$ is positive. This makes $h'()$ convex. Hence using convexity of $h'()$, second term within the braces is atleast

$$h'(q(x+\delta)^2 + (1-q)(x^2+\delta^2)) - h'(x^2)$$
$$= h'(x^2+\delta^2+2q\delta x) - h'(x^2) > 0$$

Where the last inequality follows from strictly increasing nature of $h'()$. Now the second observation implies that the LHS of inequality [4.55] is strictly negative for $q \in [0,0.5]$. Now if $x_q^*(0)' \equiv x'$ is positive anywhere in the range $q \in [0,0.5]$, then following the analysis above, RHS will be strictly positive, hence a contradiction ! Thus $m^*(0)$ decreases with $q \in [0,0.5]$. ■

## 4.4 Calculating the Optimal Cost-To-Go after Placement

We observe from (4.20) that $J_\lambda(0,0)$ is required to obtain the OSLA placement set $\overline{\mathscr{P}}_\lambda$. $J_\lambda(0,0)$ is the optimal cost-to-go after a relay is placed and the path continues. Our aim now is to compute $J_\lambda(0,0)$. To this end, for each $h \geq 0$, define

$$\mathscr{P}(h) \quad = \quad \{(m,n) \in \mathbb{Z}_+^2 : p(\lambda+h) \leq q\Delta_1(m,n) + (1-q)\Delta_2(m,n)\}. \quad (4.56)$$

This will yield a countable collection of sets, which include $\mathscr{P}_\lambda$. For each such set $\mathscr{P}$, define the placement boundary $\mathscr{B}$ as follows

$$\mathscr{B} \quad = \quad \{(m,n) \in \mathscr{P} : (m-1,n) \in \mathscr{P}^c \text{ or } (m,n-1) \in \mathscr{P}^c\} \quad (4.57)$$

where $\mathscr{P}^c := \mathbb{Z}_+^2 - \mathscr{P}$. For each $\mathscr{P}$ define $g(\mathscr{P})$ as the cost-to-go (starting from $(0,0)$) if the placement set $\mathscr{P}$ is employed. This can be computed by a renewal argument by conditioning on whether the lattice path ends before the placement boundary, and if not then where the placement boundary is hit. To this end, we need the following definitions. The boundary $\mathscr{B}$ can be written as a disjoint union of three sets $\mathscr{B}^w$, $\mathscr{B}^s$ and $\mathscr{B}^{null}$ where

$$\begin{aligned} \mathscr{B}^w &= \{(m,n) \in \mathscr{B} : (m-1,n) \in \mathscr{B}\} && (4.58) \\ \mathscr{B}^s &= \{(m,n) \in \mathscr{B} : (m,n-1) \in \mathscr{B}\} && (4.59) \\ \mathscr{B}^{null} &= \{(m,n) \in \mathscr{B} : (m-1,n) \notin \mathscr{B} \text{ and} \\ & \quad (m,n-1) \notin \mathscr{B}\}. && (4.60) \end{aligned}$$

It is easy to check that these sets are mutually disjoint and $\mathscr{B} = \mathscr{B}^w \cup \mathscr{B}^s \cup \mathscr{B}^{null}$. For a depiction of the various boundary points for the optimal boundary, see Fig. 5.4.

Note that, we can reach $(m,n) \in \mathscr{B}^{null} \cup \mathscr{P}^c$ either from West $(m-1,n)$ or South $(m,n-1)$ in the final step. However, to reach a point $(m,n) \in \mathscr{B}^s$, we must reach the point $(m-1,n)$ in the penultimate step and then take a final step in the East direction. Similarly, to reach a point $(m,n) \in \mathscr{B}^w$, we must reach the point $(m,n-1)$ in the penultimate step and then take a final step in the Northward direction. We can write down the following *reaching probabilities*. Let $\mathbb{P}((m,n),\mathsf{e})$ denote the probability of reaching the point $(m,n)$ with the path ending there and $\mathbb{P}((m,n),\mathsf{c})$ denote the probability of reaching the point $(m,n)$ with the path continuing. Hence we have,

$$\mathbb{P}((m,n),\mathsf{e}) = \begin{cases} \binom{m+n}{m} p(1-p)^{m+n-1} q^m (1-q)^n & \text{if } (m,n) \in \mathscr{P}^c \cup \mathscr{B}^{null} \\ \binom{m+n-1}{m} p(1-p)^{m+n-1} q^m (1-q)^n & \text{if } (m,n) \in \mathscr{B}^w \\ \binom{m+n-1}{m-1} p(1-p)^{m+n-1} q^m (1-q)^n & \text{if } (m,n) \in \mathscr{B}^s \end{cases}$$

Similarly we have,

$$\mathbb{P}((m,n),\mathsf{c}) = \begin{cases} \binom{m+n}{m} (1-p)^{m+n} q^m (1-q)^n & \text{if } (m,n) \in \mathscr{P}^c \cup \mathscr{B}^{null} \\ \binom{m+n-1}{m} (1-p)^{m+n} q^m (1-q)^n & \text{if } (m,n) \in \mathscr{B}^w \\ \binom{m+n-1}{m-1} (1-p)^{m+n} q^m (1-q)^n & \text{if } (m,n) \in \mathscr{B}^s \end{cases}$$

The renewal argument outlined earlier yields:

$$g_\lambda(\mathscr{P}) = \sum_{(m,n) \in \mathscr{P}^c \cup \mathscr{B}} \mathbb{P}((m,n),\mathsf{e}) d(m,n) + \sum_{(m,n) \in \mathscr{B}} \mathbb{P}((m,n),\mathsf{c})(g(\mathscr{P}) + \lambda + d(m,n)) \tag{4.61}$$

Solving for $g_\lambda(\mathscr{P})$, we obtain

$$g_\lambda(\mathscr{P}) = \frac{1}{1-\sum_{(m,n)\in\mathscr{B}}\mathbb{P}((m,n),\mathsf{c})} \times \left( \sum_{(m,n)\in\mathscr{P}^c\cup\mathscr{B}} \mathbb{P}((m,n),\mathsf{e})d(m,n) + \sum_{(m,n)\in\mathscr{B}} \mathbb{P}((m,n),\mathsf{c})(\lambda+d(m,n)) \right) . \quad (4.62)$$

It then follows that $J_\lambda(0,0)$ is obtained by minimizing $g(\mathscr{P})$ over the placement sets. From (4.20), given $J_\lambda(0,0)$, one can obtain the OSLA placement set $\overline{\mathscr{P}}_\lambda$ and thus also the optimal placement set $\mathscr{P}_\lambda$ (since $\mathscr{P}_\lambda = \overline{\mathscr{P}}_\lambda$ from Theorem 4.3.4).

As in the LOS case, it has been shown in the following section that for each $0 < p < 1$ and $\lambda \geq 0$, there exists $\overline{g}_\lambda(<\infty)$, such that $J_\lambda(0,0) \leq \overline{g}_\lambda$. In order to compute $J_\lambda(0,0)$ we need to search only over the real line segment $[0,\overline{g}_\lambda]$.

## 4.5 An Upper-Bound $\overline{g}_\lambda$ for the Optimal Cost-to-Go $J_\lambda(0,0)$

Since cost-to-go for *any* policy is atleast equal to the optimal cost-to-go $J_\lambda(0,0)$, we may upper bound $J_\lambda(0,0)$ by the cost-to-go for *any* policy. Now consider the policy $\mathscr{P}'$ in which we place a node at each step until the lattice-path ends. If the cost-to-go for the policy $\mathscr{P}'$ be denoted by $\overline{g}_\lambda$, then we can write the following renewal theoretic equation for $\overline{g}_\lambda$

$$\begin{aligned} \overline{g}_\lambda &= \lambda + pd(1) + (1-p)(\lambda + d(1) + \overline{g}_\lambda) \\ \overline{g}_\lambda &= \frac{1}{p}((2-p)\lambda + d(1)) \end{aligned} \quad (4.63)$$

Then it follows that

$$J_\lambda(0,0) \leq \frac{1}{p}((2-p)\lambda + d(1)) \quad (4.64)$$

## 4.6 An Efficient Fixed Point Iteration Algorithm for Obtaining the Optimal Policy

As in the LOS case, we now present a fixed point iteration algorithm based on the OSLA rule 4.20. The notations are similar to those used in the LOS case in section 3.4.

In the following, we first derive an equation which is satisfied by any threshold policy of the form (4.56). Then we prove a series of lemmas to establish the correctness and finite convergence of Algorithm 2.

**Algorithm 2** Calculate $J(0,0)$ and $\mathscr{P}_\lambda$

**Require:** $0 < p < 1$, $0 \leq q \leq 1$, $\lambda \geq 0$

$$h \leftarrow 0$$

**while** 1 **do**

$$\mathscr{P}_h \leftarrow \{(m,n) \in \mathbb{Z}_+^2 : p(\lambda + h) \leq \Delta(m,n)\}$$

$$\mathscr{B}_h \leftarrow \{(m,n) \in \mathscr{P}_h : (m-1,n) \in \mathscr{P}_h^c, \text{ Or,} (m,n-1) \in \mathscr{P}_h^c\}$$

$$g_h \leftarrow \frac{1}{1 - \sum_{(m,n)\in\mathscr{B}_h} \mathbb{P}((m,n),\mathsf{c})} \left( \sum_{(m,n)\in\mathscr{P}_h^c \bigcup \mathscr{B}_h} (\mathbb{P}((m,n),\mathsf{e}) d(m,n) + \sum_{(m,n)\in\mathscr{B}_h} \mathbb{P}((m,n),\mathsf{c})(\lambda + d(m,n)) \right)$$

  **if** $g_h == h$ **then**

    break;

  **end if**

  $h \leftarrow g_h$

**end while**

**return** $g_h$, $\mathscr{P}_h$

From Eqn 4.61 we have,

$$g(h) = \sum_{(m,n)\in\mathscr{P}^c(h)\cup\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{e}) d(m,n) + \sum_{(m,n)\in\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{c})(g(h) + \lambda + d(m,n))$$

Now we introduce some notations.

- A path $\pi$ is a walk through the integer lattice, starting from the origin $(0,0)$. If the point $(m,n)$ is on the path $\pi$, the path $\pi$ either stops at the next point (w.p. $p$) or continues atleast one more step further (w.p. $1-p$). In either case, the next point on the path is $(m+1,n)$ w.p. $q$ or, $(m,n+1)$ w.p. $(1-q)$. The set of all paths is denoted by $\Pi$. The set of all paths that end at the point $(m,n)$ is denoted by $\Pi_{mn}$, $(m,n) \in \mathscr{P}^c(h) \bigcup \mathscr{B}(h)$. The set of path that continues beyond the boundary $\mathscr{B}(h)$ is denoted by $\Pi(c) = \Pi - \bigcup_{(m,n)\in\mathscr{P}^c(h)\bigcup\mathscr{B}(h)} \Pi_{mn} = \biguplus_{(m,n)\in\mathscr{B}(h)} \Pi_{mn}(c)$. Where $\Pi_{mn}(c)$ denotes the set of paths that reaches at the point $(m,n) \in \mathscr{B}(h)$ and continues. The symbol $\uplus$ denotes disjoint union.

- Let us denote the set of edges whose both the end vertices belongs to the set $\mathscr{P}^c(h) \bigcup \mathscr{B}(h)$ by $E$. Consider a path $\pi \in \Pi$. It is completely characterized by its edge set $E_\pi$.

- Reaching probability $r(m,n)$ of a point $(m,n) \in \mathscr{P}^c(h) \bigcup \mathscr{B}(h)$ is defined as the probability that a random path $\pi \in \Pi$ *reaches* the point $(m,n)$, without the path ending at that point. Clearly $r(m,n) = \binom{m+n}{m}(1-p)^{m+n} q^m (1-q)^n$

- The incremental cost function $\delta : E \longrightarrow \mathbb{R}_+$ is defined as follows

$$\delta(e) = d(m+1,n) - d(m,n) = \Delta_1(m,n) \qquad \text{if} \quad e = \{(m,n),(m+1,n)\} \tag{4.65}$$
$$= d(m,n+1) - d(m,n) = \Delta_2(m,n) \qquad \text{if} \quad e = \{(m,n),(m,n+1)\} \tag{4.66}$$

The incremental cost function allows us to write

$$d(m,n) = \sum_{e \in E_\pi} \delta(e) + d(0,0) \tag{4.67}$$

The above expression is valid for all $\pi \in \Pi_{mn}$ if $(m,n) \in \mathscr{P}^c(h) \bigcup \mathscr{B}(h)$ and $\pi \in \Pi_{mn}(c)$ if $(m,n) \in \mathscr{B}(h)$
Now consider

$$\begin{aligned}
& \sum_{(m,n)\in\mathscr{P}^c(h)\cup\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{e})d(m,n) + \sum_{(m,n)\in\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{c})d(m,n) \\
= & \sum_{(m,n)\in\mathscr{P}^c(h)\cup\mathscr{B}(h)} \sum_{\pi\in\Pi_{mn}} p(\pi)\left(\sum_{e\in E_\pi}\delta(e)+d(0,0)\right) + \sum_{(m,n)\in\mathscr{B}(h)} \sum_{\pi\in\Pi_{mn}(c)} p(\pi) \sum_{e\in E_\pi\cap E}\left(\delta(e)+d(0,0)\right) \\
= & \sum_{e\in E}\delta(e)\sum_{\pi\in\Pi: e\in E_\pi} p(\pi) + d(0,0) \\
= & \sum_{e\in E}\delta(e)p(e) + d(0,0)
\end{aligned} \tag{4.68}$$

Where by $p(e)$ we denote the probability that a random path continues through the edge $e \in E$.
Now if $e$ is horizontal i.e. $e = \{(m,n),(m+1,n)\}, (m,n) \in \mathscr{P}^c(h)$, we have $p(e) = qr(m,n)$ and $\delta(e) = \Delta_1(m,n)$. Similarly if $e$ is vertical i.e. $e = \{(m,n),(m,n+1)\}, (m,n) \in \mathscr{P}^c(h)$, we have $p(e) = (1-q)r(m,n)$ and $\delta(e) = \Delta_2(m,n)$. Using these relations, we may rewrite Eqn 4.68 as follows

$$\begin{aligned}
& \sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Big(q\Delta_1(m,n)+(1-q)\Delta_2(m,n)\Big) + d(0,0) \\
= & \sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Delta(m,n) + d(0,0)
\end{aligned} \tag{4.69}$$

Now consider the probability $\sum_{(m,n)\in\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{c})$. It is the probability that a random path continues beyond the boundary $\mathscr{B}(h)$. Hence we may write

$$\begin{aligned}
& \sum_{(m,n)\in\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{c}) \\
= & \; 1 - \sum_{(m,n)\in\mathscr{P}^c(h)\bigcup\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{e}) \\
= & \; 1 - \sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)p
\end{aligned} \tag{4.70}$$

Now we are ready to derive an equation similar to Eqn 3.67 as in the LOS case. From Eqn 4.61 we have

$$\begin{aligned}
g(h) &= \sum_{(m,n)\in\mathscr{P}^c(h)\cup\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{e})d(m,n) + \sum_{(m,n)\in\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{c})(g(h)+\lambda+d(m,n)) \\
&= \left(\sum_{(m,n)\in\mathscr{P}^c(h)\cup\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{e})d(m,n) + \sum_{(m,n)\in\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{c})d(m,n)\right) + (\lambda+g(h))\left(\sum_{(m,n)\in\mathscr{B}(h)} \mathbb{P}((m,n),\mathsf{c})\right) \\
&= \sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Delta(m,n) + d(0,0) + (\lambda+g(h))\left(1-\sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)p\right) \\
&= \sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Big(\Delta(m,n)-p(\lambda+g(h))\Big) + d(0,0)+\lambda+g(h)
\end{aligned}$$

Where we have used Eqns 4.69 and 4.70 in the above derivation. Simplifying the above expression, we get the following equation which is satisfied by *any* placement policy of the form 4.56.

$$0 = \sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Big(\Delta(m,n)-p(\lambda+g(h))\Big) + d(0,0)+\lambda \tag{4.71}$$

We now prove the following lemma.

**Lemma 4.6.1.** *If $h > g^*$ then $h > g(h)$.*

*Proof.* We recall the definition of $\mathscr{P}^c(h)$.

$$\mathscr{P}^c(h) = \{(m,n)\in\mathbb{Z}_+^2 : p(\lambda+h) > \Delta(m,n)\} \tag{4.72}$$

Since $h > g^*$, we immediately conclude that $\mathscr{P}^{*c} \subset \mathscr{P}^c(h)$.

From Eqn 4.71, we may write the following expression for the optimal policy $\mathscr{P}^*$

$$\sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n)\Delta(m,n) = p(\lambda+g^*)\left(\sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n)\right) - \Big(d(0,0)+\lambda\Big) \tag{4.73}$$

We may similarly write the following expression for the policy $\mathscr{P}(h)$

$$\sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Delta(m,n) = p(\lambda+g(h))\left(\sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\right) - \Big(d(0,0)+\lambda\Big) \tag{4.74}$$

Now since $\mathscr{P}^{*c} \subset \mathscr{P}^c(h)$, we may expand the LHS of 4.74 as follows

$$\begin{aligned}
& \sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Delta(m,n) && (4.75)\\
= & \sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n)\Delta(m,n) + \sum_{(m,n)\in\mathscr{P}^c(h)-\mathscr{P}^{*c}} r(m,n)\Delta(m,n) && (4.76)\\
< & \sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n)\Delta(m,n) + p(\lambda+h)\Big(\sum_{(m,n)\in\mathscr{P}^c(h)-\mathscr{P}^{*c}} r(m,n)\Big) && (4.77)\\
= & \; p(\lambda+g^*)\Big(\sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n)\Big) - \Big(d(0,0)+\lambda\Big) + p(\lambda+h)\Big(\sum_{(m,n)\in\mathscr{P}^c(h)-\mathscr{P}^{*c}} r(m,n)\Big) && (4.78)
\end{aligned}$$

Where in Eqn. 4.77 we have used equation 4.72 and in Eqn. 4.78, we have substituted the value for the quantity from Eqn 4.73.

Using Eqn 4.74, we may alternatively write the LHS of 4.74 as follows

$$\begin{aligned}
& \sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Delta(m,n)\\
= & \; p(\lambda+g(h))\Big(\sum_{(m,n)\in\mathscr{P}^c(h)} r(m,n)\Big) - \Big(d(0,0)+\lambda\Big)\\
= & \; p(\lambda+g(h))\Big(\sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n) + \sum_{(m,n)\in\mathscr{P}^c(h)-\mathscr{P}^{*c}} r(m,n)\Big) - \Big(d(0,0)+\lambda\Big) && (4.79)
\end{aligned}$$

Now comparing Eqns 4.78 and 4.79, we may write

$$p(\lambda+g(h))\Big(\sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n) + \sum_{(m,n)\in\mathscr{P}^c(h)-\mathscr{P}^{*c}} r(m,n)\Big) < p(\lambda+g^*)\Big(\sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n)\Big) + p(\lambda+h)\times \sum_{(m,n)\in\mathscr{P}^c(h)-\mathscr{P}^{*c}} r(m,n)$$

Rearrenging the above inequality, we get

$$p(g(h)-g^*)\Big(\sum_{(m,n)\in\mathscr{P}^{*c}} r(m,n)\Big) < p(h-g(h))\Big(\sum_{(m,n)\in\mathscr{P}^c(h)-\mathscr{P}^{*c}} r(m,n)\Big) \quad (4.80)$$

But we know that $g^* = \inf g(h)$. Hence $g(h) - g^* \geq 0$. Thus from the above inequality, we conclude that

$$h > g(h) \quad (4.81)$$

■

**Lemma 4.6.2.** *$h^{(k)} \geq g^*$ for $k \geq 1$.*

*Proof.* We have $h^{(k)} \stackrel{(1)}{=} g(h^{(k-1)}) \stackrel{(2)}{\geq} g^*$.

Where $(1)$ follows directly from successive iterations of the algorithm and $(2)$ follows from the fact that $g^*$ is the infimum of $g(\cdot)$. ■

**Corollary 4.6.3.** *$g(h)$ has a unique fixed point.*

*Proof.* We know that $g(h)$ has a fixed point at $h = g^*$. Now consider the following cases

**Case 1:** $h > g^*$

In this case, from lemma 4.6.1 we know that $h > g(h)$. Hence $g(\cdot)$ can not have a fixed point which is strictly greater that $g^*$.

**Case 2:** $h < g^*$

In this case we have $h < g^* \overset{(1)}{\leq} g(h)$. Where $(1)$ follows from the fact that $g^* = \inf g(\cdot)$. Hence in this case we have $h < g(h)$. Thus $g(\cdot)$ can not have a fixed point which is strictly less than $g^*$.

Combining the above cases the proof follows. ■

**Lemma 4.6.4.** *The sequence $\{h^{(k)}\}_{k\geq 1}$ is non-increasing i.e. $h^{(k+1)} \leq h^{(k)}$, with the equality sign holding iff $h^{(k)} = g^*$.*

*Proof.* We have for all $k \geq 1$, $h^{(k+1)} \overset{(1)}{=} g(h^{(k)}) \overset{(2)}{\leq} h^{(k)}$

Here (1) follows from successive iterations of the fixed point iterations and (2) follows from lemma 4.6.2 and 4.6.1. Clearly the equality sign in the lemma holds iff equality in (2) holds, i.e. iff $h^{(k)}$ is a fixed point of $g(\cdot)$. Then from corollary 4.6.3 it follows that $h^{(k+1)} = h^{(k)}$ iff $h^{(k)} = g^*$.

■

**Lemma 4.6.5.** *The sequence $\{\mathscr{P}^c(h^{(k)})\}_{k\geq 1}$ is non-increasing i.e. $\mathscr{P}^c(h^{(k+1)}) \subset \mathscr{P}^c(h^{(k)})$, where the containment is proper unless $\mathscr{P}^c(h^{(k)}) = \mathscr{P}^{c*}$.*

*Proof.* Recall that

$$\mathscr{P}^c(h) = \{(m,n) \in \mathbb{Z}_+^2 : p(\lambda + h) > \Delta(m,n)\}$$

Now take $h_2 > h_1$. We have

$$\begin{aligned} \mathscr{P}^c(h_1) &= \{(m,n) \in \mathbb{Z}_+^2 : p(\lambda + h_1) > \Delta(m,n)\} && (4.82)\\ &\subset \{(m,n) \in \mathbb{Z}_+^2 : p(\lambda + h_2) > \Delta(m,n)\} && (4.83)\\ &= \mathscr{P}^c(h_2) && (4.84) \end{aligned}$$

Thus from Lemma 4.6.4 it follows that the sequence $\{\mathscr{P}^c(h^{(k)})\}_{k\geq 1}$ is non-increasing.

Also $\mathscr{P}^c(h^{(k+1)}) = \mathscr{P}^c(h^{(k)})$ only if $g(h^{(k+1)}) = g(h^{(k)}) = h^{(k+1)}$. Hence from Lemma 4.6.3, it follows that $h^{(k+1)} = g^*$, and hence $\mathscr{P}^c(h^{(k)}) = \mathscr{P}^{c*}$. ■

Now we prove the main theorem in this section.

**Theorem 4.6.6.** *Algorithm 2 returns with $g^*$ and $\mathscr{P}^{c*}$ in finite time.*

Table 4.1: Iterations required for Algorithm 2 to converge for different $p$'s and $q$'s ($\eta = 3$)

| $p$ | $q$ | iterations |
|---|---|---|
| 0.002 | 0.5 | 6 |
| 0.002 | 0.3 | 6 |
| 0.002 | 0.1 | 6 |
| 0.02 | 0.5 | 5 |
| 0.02 | 0.3 | 5 |
| 0.02 | 0.1 | 5 |
| 0.2 | 0.5 | 3 |
| 0.2 | 0.3 | 3 |
| 0.2 | 0.1 | 3 |

*Proof.* From Lemma 4.6.5, it follows that starting with a finite set $\mathscr{P}^c(h^{(1)})$, the sequence of sets $\{\mathscr{P}^c(h^{(k)})\}_{k\geq 1}$ decreases strictly at each iteration until it reaches the optimum $\mathscr{P}^{c*}$. At this point we have $h^{(k)} = g(h^{(k)}) = g^*$ and the algorithm terminates. ■

## 4.7 Solving the Constrained MDP

In Section 4.4 we devised a procedure to obtain an optimal placement set $\mathscr{P}_\lambda$, using which the expected number of relays used by the optimal policy can be computed as

$$\mathbb{E}_{\pi_\lambda} N = \frac{\sum_{(m,n)\in\mathscr{B}_\lambda} \mathbb{P}((m,n),\mathsf{c})}{1 - \sum_{(m,n)\in\mathscr{B}_\lambda} \mathbb{P}((m,n),\mathsf{c})} \tag{4.85}$$

where $\mathbb{P}((m,n),\mathsf{c})$ is the reaching probability corresponding to $\mathscr{P}_\lambda$. A plot of $\mathbb{E}_{\pi_\lambda} N$ vs. $\lambda$ is given in Fig. 5.2.

Recalling the discussion in Section 3.5, the solution to the constrained problem defined in Eqn. (2.2) can be obtained using Theorem 3.5.1. As in Part (ii)-(b) of Theorem 3.5.1, whenever $\rho_{avg} < \rho_{max}$, if there is a $\lambda$ such that $\underline{\rho} < \rho_{avg} < \overline{\rho}$, then one has to now mix between two optimal boundaries (unlike for the LOS case where we mix between two thresholds) to obtain a solution for the constrained problem.

## 4.8 Summary

In this chapter, we studied the problem of impromptu relay deployment along a random corridor with NLOS propagation. In section 4.1, we formulated the problem as an Infinite Horizon Total Cost MDP. In section 4.2.1, we solved the relaxed problem and showed that the optimal policy is in the form of a two-dimensional threshold policy. In section 4.3, we posed the problem as an optimal stopping problem and showed the equivalence of optimal policy with the OSLA policy. In section 4.6 we proposed an efficient fixed-point-iteration algorithm to solve for the optimal policy exactly and showed its correctness and finite termination properties.

# Chapter 5

# Numerical Results

In numerical work, we take the one-hop power function $d(r) = P_m + \gamma r^\eta$, where $P_m = 0.1, \gamma = 0.01$ and $\eta = 2$, unless otherwise specified.

In Figure 3 we provide a set of numerical results for the common parameter value $p = 0.002$.

In Figure 5.2 we plot $E_{\pi_\lambda} N$ and $E_{\pi_\lambda} C$ vs. $\lambda$. Since $\lambda$ is the cost per relay, as expected, $E_{\pi_\lambda} N$ decreases as $\lambda$ increases. We observe that $E_{\pi_\lambda} C$ and the optimal total cost $J_\lambda(0,0)$ increase as $\lambda$ increases. A close examination of Figure 3(b) reveals that both the plots are step functions. This is due to the discrete placement at lattice points, which results in the same placement boundary being optimal for a range of $\lambda$ values. Thus, as seen in Section 3.5, at the $\lambda$ values where there is jump in $E_{\pi_\lambda} N$, a random mixture of two policies is needed.

The plots in Figures 5.1 and 5.2 were for $q = 0.5$. Figure 5.3 shows the variation of the total optimal cost $J_\lambda(0,0)$ with $q$. The variation is symmetric about $q = 0.5$ For a given path length, $q = 0.5$ results in the path folding frequently. In such a case, since NLOS propagation is permitted, and the path loss is isotropic, fewer relays are required to be placed. On the other hand, when $q$ is close to 0 or to 1 the path takes fewer turns and more relays are needed, leading to larger values of the total cost.

In Figure 5.4 we show an optimal placement boundary for $p = 0.002, q = 0.5$, and $\eta = 3$. Since $q = 0.5$ the boundary is symmetric about the $m = n$ line. The various sets of boundary points defined in Section 4.4 are shown in different colours in Figure 3(d). In Figure 5.5 we show the variation of optimal boundaries with $\eta$. As $\eta$, the path loss exponent, increases the hop cost increases for a given hop distance. This results in relays needing to be placed more frequently. As can be seen the placement boundaries shrink with increasing $\eta$. We also notice that the placement boundary for $\eta = 2$ is a straight line; indeed this provable result holds for $\eta = 2$ for any values of $p$ and $q$.

## 5.1 Comparison Between a Simple and the Optimal Policy

In this section we propose a simple policy and compare its performance with the optimal policy.

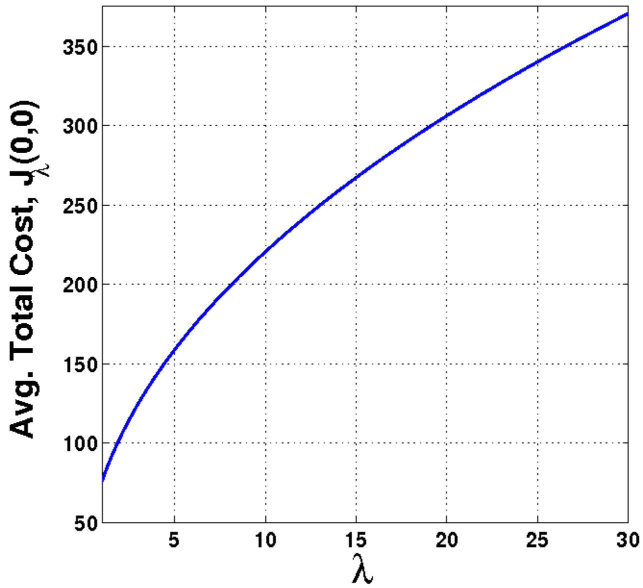



Figure 5.1: Variation of Optimal Cost-to-go with 'Relay Price' $\lambda$

### 5.1.1 The *Simple* Policy

We recall from the literature survey in Section 1.0.1 that prior literature invariably proposed the policy of placing a relay after the RF signal strength from the previous relay has dropped below a threshold. For isotropic propagation (as we have assumed in this thesis), this is equivalent to placing the relay after a circular boundary is crossed. With this in mind, we obtained the *simple optimal constant distance placement policy* [1]numerically (in a manner similar to what is described in Section 4.4). More specifically we now consider the placement set to be

$$\mathscr{P}^h(\alpha) = \{(m,n) \in \mathbb{Z}_+^2 : m^2 + n^2 \geq \alpha^2\} \qquad \alpha \in \mathbb{R}_+$$

From the very definition (5.1) of $\mathscr{P}^h(\alpha)$, it is clear that it is a threshold policy. Hence we may similarly define

$$\begin{aligned} \mathscr{B}^h(\alpha) \quad &= \quad \{(m,n) \in \mathscr{P}(\alpha) : (m-1,n) \in \mathscr{P}^c(\alpha) \text{ or} \\ &\qquad (m,n-1) \in \mathscr{P}^c(\alpha)\} \end{aligned} \tag{5.1}$$

[1]Here we use the words "simple" and "heuristic" interchangably.

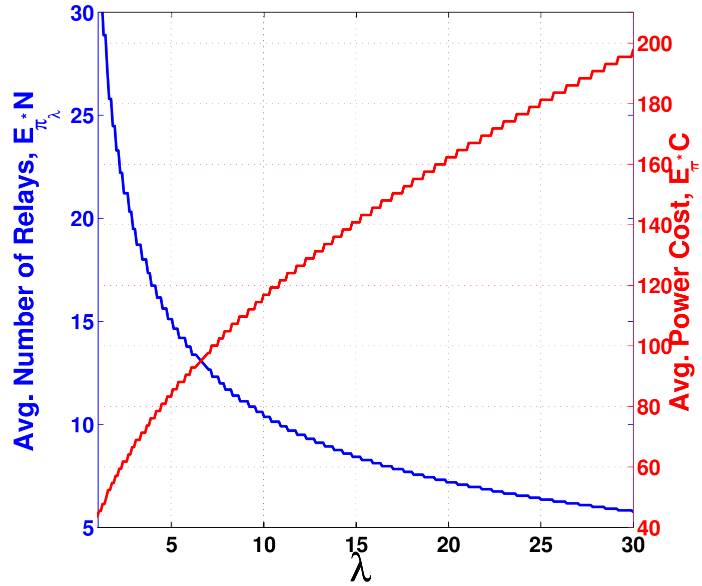



Figure 5.2: Variation of Expected Cost and Expected Number of Relay nodes used with Relay Price $\lambda$

where $\mathscr{P}^{h^c}(\alpha) := \mathbb{Z}_+^2 - \mathscr{P}^h(\alpha)$.

And the cost-to-go for this policy $\mathscr{P}^h(\alpha)$ can be calculated as

$$g_\lambda(\mathscr{P}^h(\alpha)) = \frac{1}{1-\sum_{(m,n)\in\mathscr{B}^h(\alpha)}\mathbb{P}((m,n),\mathsf{c})}\left(\sum_{(m,n)\in\mathscr{P}^{h^c}(\alpha)\cup\mathscr{B}^h(\alpha)}\mathbb{P}((m,n),\mathsf{e})d(m,n)+\sum_{(m,n)\in\mathscr{B}^h(\alpha)}\mathbb{P}((m,n),\mathsf{c})(\lambda+d(m,n))\right).$$

Subsequently $g_\lambda(\mathscr{P}^h(\alpha))$ is minimized over $\alpha$ to get the optimal cost-to-go among this class of policies.

A sample result is provided in Figure 5.6, for the parameters $p = 0.002, q = 0.5, \eta = 2$. We observe that if the path were to evolve roughly Eastward or Northward then the constant distance placement will result in many more relays being placed. On the other hand if the path evolves diagonally (which has higher probability) then the two placement boundaries will result in similar placement decisions. This observation shows up in Figure 5.7 where we show the optimal cost for the constrained problem, with the constraint $\rho$ being plotted on the horizontal axis. We find that for $q = 0.5$ the optimal placement boundary and the optimal constant distance placement provide costs that are almost indistinguishable at this scale. In this plot we also show the optimal cost for $q = 1$ (i.e., straight line path); as expected, the cost is much larger since the path does not fold.

We have plotted the optimal policies (the placement boundaries, to be specific) corresponding to the same $\rho$ values for different choices of the parameters $p$ and $q$. The figures are shown below. Red and blue points denote boundary points in optimal and heuristic policies respectively. It can be observed that two policies match quite well in the region where a random path with parameters $p$ and $q$ has a high reaching probability. The cost incurred using these two policies also have been plotted subsequently and the curves are found to agree quite well. This explains the reason why the simple policy performs quite close to the optimal policy.

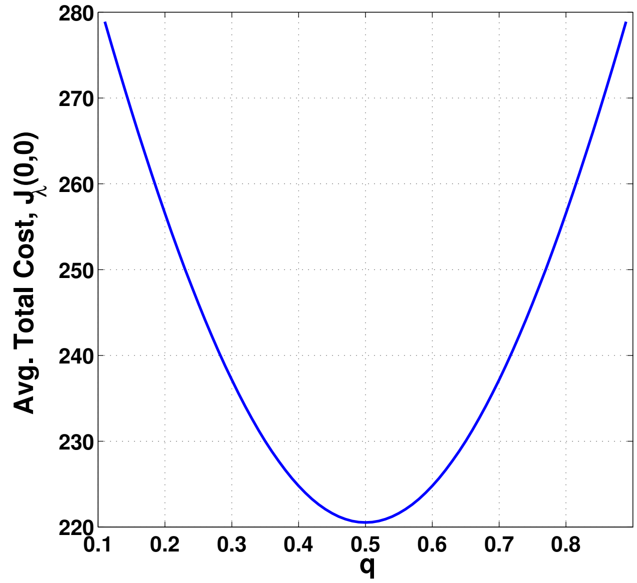



Figure 5.3: Symmetric Variation of Cost-to-go about $q = 1/2$

However as shown in Figure 5.10(d), if we take a *Parameter-agnostic-policy*, i.e. we use a fixed constant distance policy for different parameter values $p, q$, it is clear from the figure that the cost-to-go $J(0,0)$ for such a policy for a fixed $\lambda$ is quite larger as compared to the optimal policy. Hence the threshold radius for the *simple* policy must be adaptively changed with $p$ and $q$ to get a near-optimal performance.

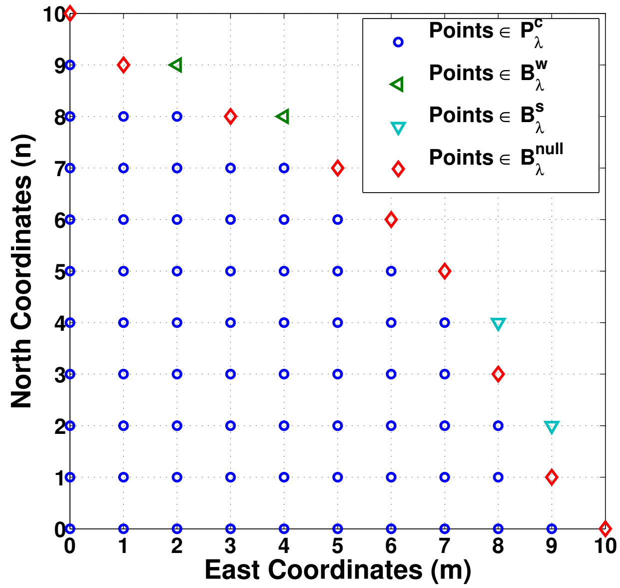



Figure 5.4: Complement of the Placement Set and Various Components of the Boundary ($p = 0.002, q = 1/2, \eta = 3, \rho = 10$)

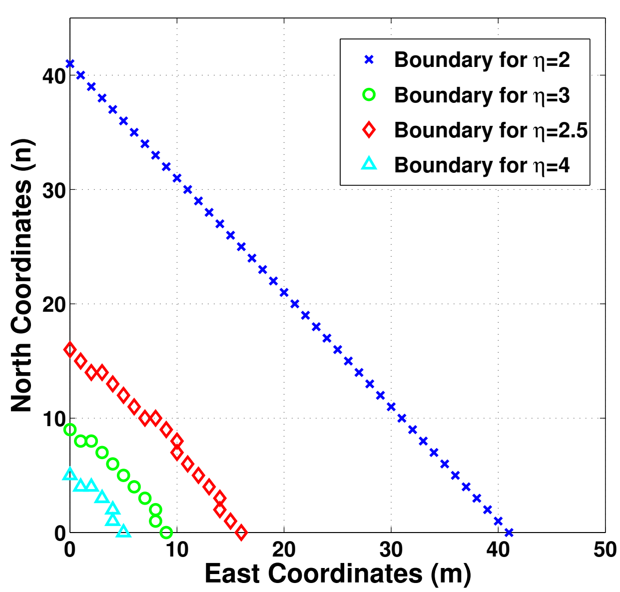



Figure 5.5: Variation of Placement boundary with path-loss-exponent $\eta$

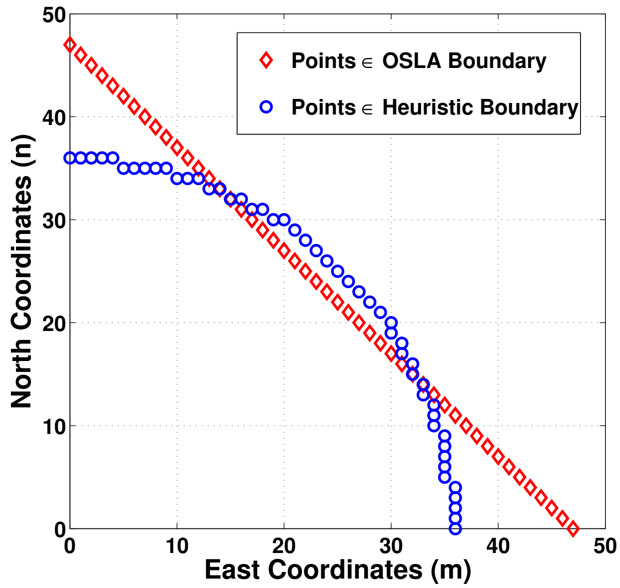



Figure 5.6: Comparison between the Optimal and Heuristic placement policies ($p = 0.002, q = 0.5, \rho = 10$)

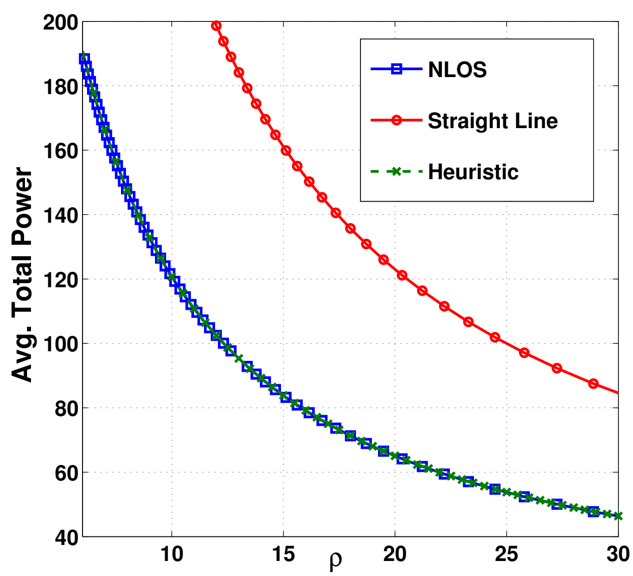
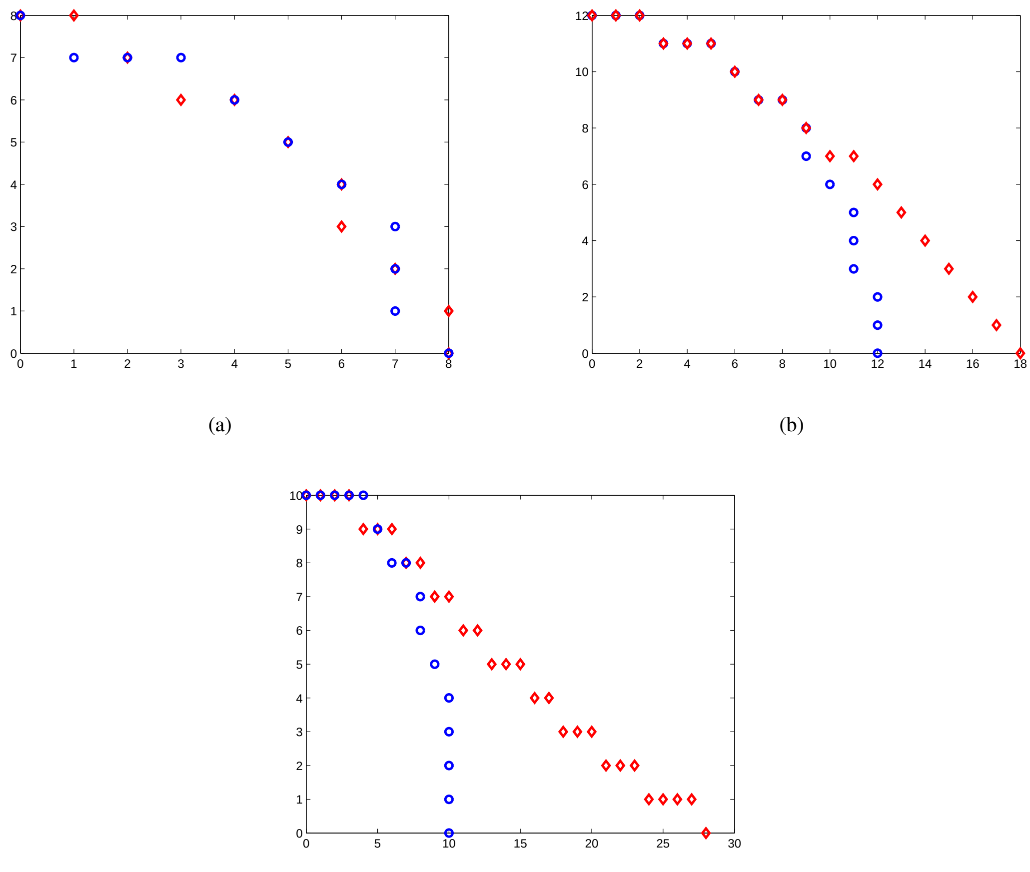



Figure 5.7: Comparison of Avg Power costs vs $\rho$ curves for three different cases ($p = 0.002, q = 0.5, \eta = 2$)



(a) (b) (c)

Figure 5.8: Boundary Comparison for $p = 0.02, \eta = 2$ and (a) $q = 0.5, \rho = 4.7$. (b) $q = 0.3, \rho = 2.8$. (c) $q = 0.1, \rho = 4.0$. The red diamonds denote boundary points in Optimal Policy and blue circles denote boundary points in Heuristic Policy

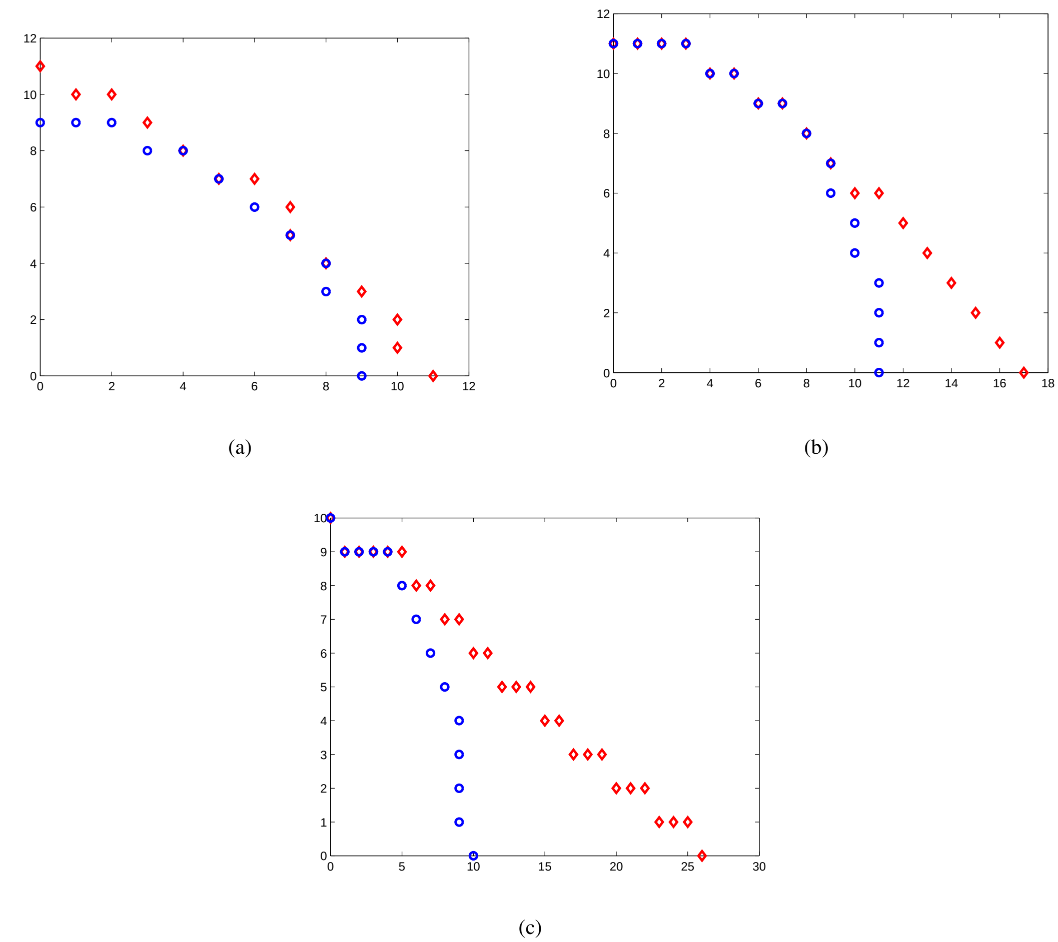



Figure 5.9: Boundary Comparison for the parameters $p = 0.2, \eta = 2$ and (a) $q = 0.5, \rho = 0.07$. (b) $q = 0.3, \rho = 0.045$. (c) $q = 0.1, \rho = 0.11$. The red diamonds denote boundary points in Optimal Policy and blue circles denote boundary points in Heuristic Policy

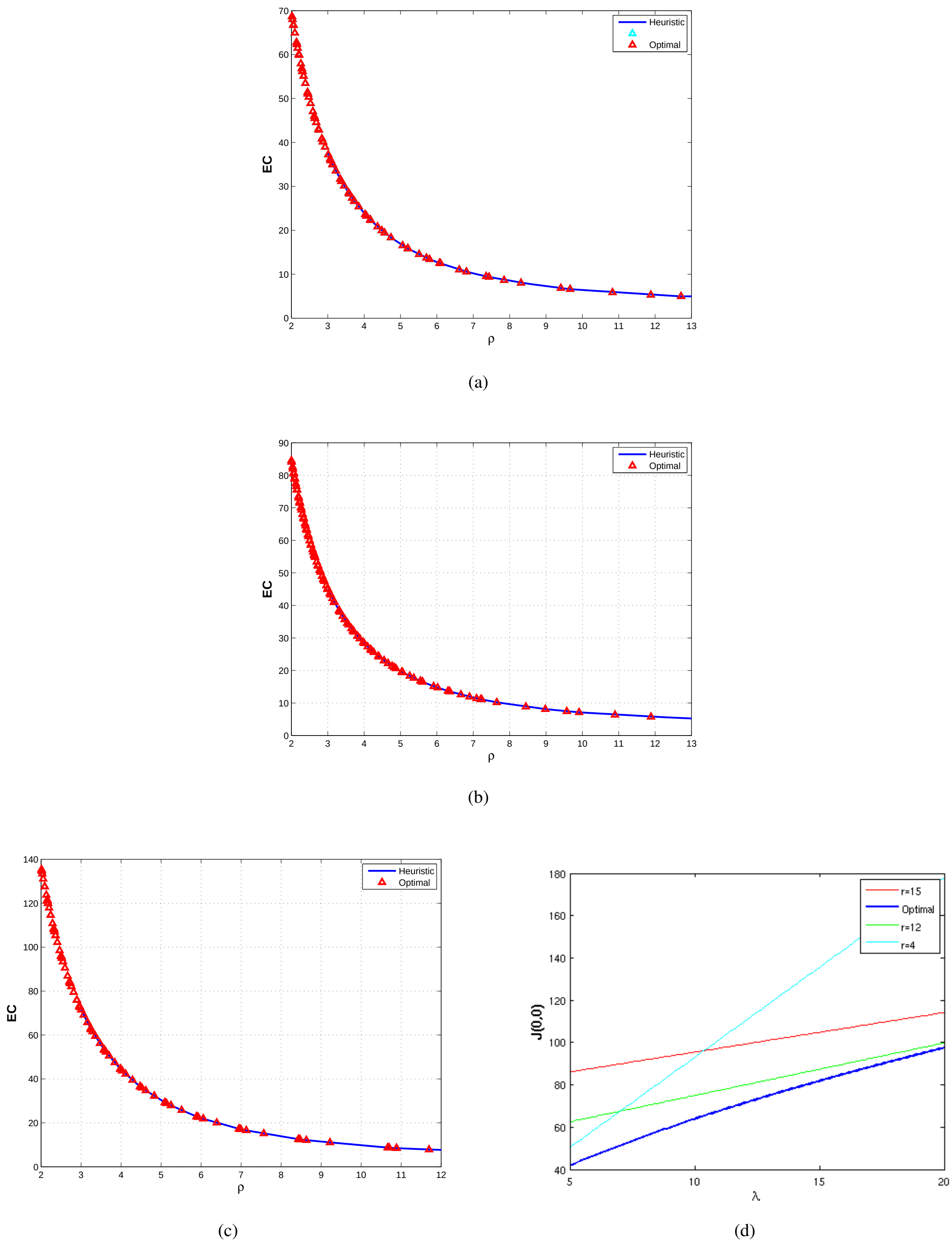
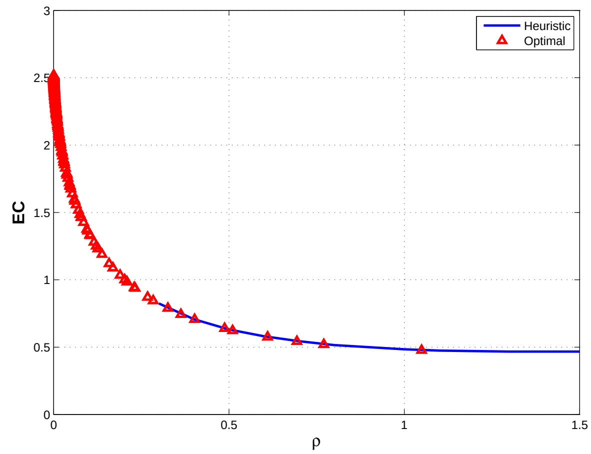
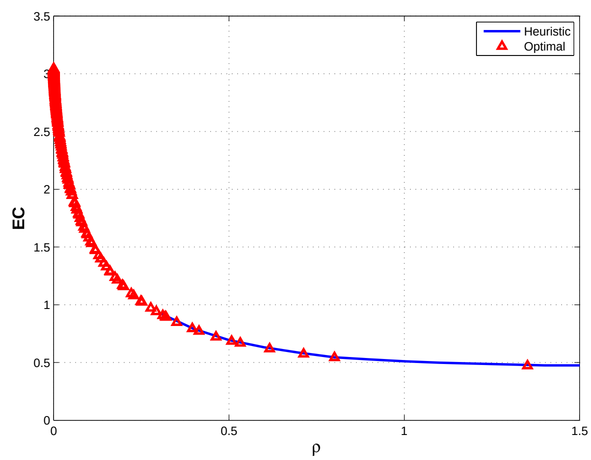


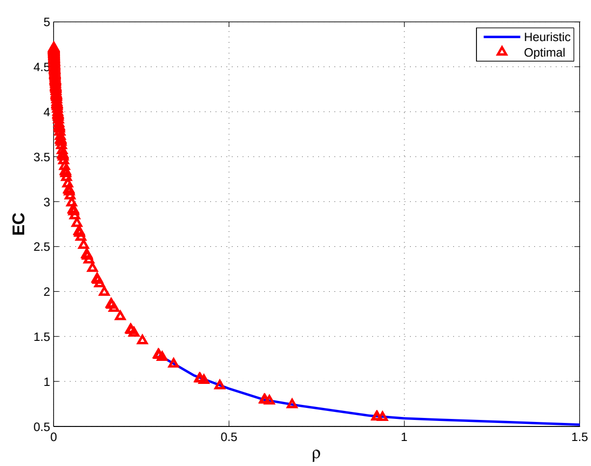

Figure 5.10: Comparison of Power-Cost between the Optimal and the Simple Placement policies for $p = 0.02$ and (a) $q = 0.5$ (b) $q = 0.3$ (c) $q = 0.1$ (d)Comparison of Cost-to-go between the Optimal and three other constant-distance policies for the relaxed problem with $\lambda = 10, p = 0.02, q = 0.5$.




(a)




(b)




(c)

Figure 5.11: Comparison of Cost-to-go between the Optimal and the Heuristic Placement policies for $p = 0.2$ and (a) $q = 0.5$ (b) $q = 0.3$ (c) $q = 0.1$

# Chapter 6

# Conclusion and Future Work

We considered the problem of placing relays on a random lattice path to optimize a linear combination of average power cost and the average number of relays deployed. Two different propagation models, namely, LOS and NLOS, were studied. The optimal placement policies for both the models were proved to be of threshold nature (single threshold for the LOS case, Theorem 3.2; threshold boundary in the case of NLOS, Theorem 4.2). We further proved the optimality of the OSLA rule (in Theorem 3.4 and 4.6, for LOS and NLOS cases, respectively). We have developed an efficient fixed-point-iteration algorithm to compute the optimal policy for both the LOS and NLOS cases. Through numerical work we observed that the performance (in terms of average power incurred for a given relay constraint) of the best distance threshold policy is close to that of the optimal.

Our future work will comprise of extending the theoretical results obtained here to the scenario where we have access to measurement of the time-varying wireless channel as we continue deploying. We also wish to extend our results to more complicated deployment region than simple straight line segments as considered in this thesis.

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