Information Theory Meets Percolation Theory : Capacity Scaling in Wireless Relay Network

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Abstract

In this expository article we revisit the topic of capacity scaling in wireless relay networks from a percolation theory perspective. In particular, following the works of Franceschetti et al [2]., we give an outline of a construction to show that in a large wireless network with interference, it is possible, w.h.p., to sustain a non-zero throughput between *any* two source-destination pair chosen among any α fraction of nodes, $0 < \alpha < 1$. Then using Information Theoretic arguments and in particular, utilizing the Broadcast-Cut bound from multi terminal network information theory, we prove a converse result and show that w.h.p., there exists a set of strictly positive fraction of nodes such that it is impossible to sustain any given non-zero throughput between any source-destination pair in it. In this article, our contribution is to develop a new, short and simple yet rigorous proof for the converse utilizing *Monotone Convergence Theorem* and *Campbell's Theorem*.

1 Introduction

In this article we consider asymptotic source-destination multihop relay capacity in a static wireless network, where a large number of wireless nodes are distributed in a plane according to a Poisson Point Process of unit intensity. We assume an underlying multi terminal AWGN channel, with a given loss function $l()$, which satisfies some regularity conditions. Under a similar network model, the following important result was established in [3], where $\lambda(n)$ denote a feasible throughput that *every* node is able to deliver to its destination.

Theorem 1. *(Main result 4 in [3]) There exists constants c and c' s.t.*

$$\lim_{n\to\infty} \mathbb{P}(\lambda(n) = \frac{c}{\sqrt{n \log n}} \text{ is feasible }) = 1 \quad (1)$$

$$\lim_{n\to\infty} \mathbb{P}(\lambda(n) = \frac{c'}{\sqrt{n}} \text{ is feasible}) = 0 \quad (2)$$

This indicates that $\lambda(n)$ asymptotically converges to zero. In this article, we look at a slightly different situation by fixing source s and destination d and assuming that only s has packets to send. Other nodes have no packets to send, and they potentially help in $s-d$ transmission, by possibly relaying some packets belonging to the $s - d$ session. We show that a non-zero asymptotic throughput can be achieved w.h.p. for a given α fraction of the network. We also prove a converse result, by invoking information theoretic ideas to show that positive throughput can not be achieved among the entire network w.h.p.

2 System Model

2.1 Network Model

In this article we consider realizations of an *extended network* which is formed by placing nodes according to a Poisson Point Process of unit intensity in the plane $\mathbb{R}^2$ and restricting them to the square $B_n = [-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}] \times [-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}]$ w.p. 1. This ensures an expected number of n nodes within the square B_n. Now, for any $r > 0$, we impose a *Boolean* connectivity model on B_n by connecting any two nodes in B_n iff their euclidean distance is less than or equal to r. We call the resulting graph $G_n(r)$. We then fix a suitable value of r and study the random graph families $\{G_n(r)\}$ by letting $n \to \infty$ to obtain single-source-single-destination asymptotic throughput scaling results.

2.2 Interference Model

We assume that any of the n users, if scheduled, transmits with power P. For the achievability proof, we choose the simplest one-hop pairwise coding-decoding strategy and a TDM schedule for transmission. The pair-wise coding scheme has the advantage of being practically implementable with low overhead and decoding complexity. Decoding is done at receiver by simply treating other simultaneous transmissions as noise. Hence, assuming an AWGN channel with interference, an *achievable* rate for transmission from node i to node j over unit bandwidth is given by

$$R(x_i, x_j) = \log\left(1 + \frac{Pl(x_i, x_j)}{N_0 + \sum_{k \neq i} l(x_k, x_i)}\right) \tag{3}$$

Where $l(x_i, x_j)$ denotes path loss between x_i and x_j and is given by

$$l(x_i, x_j) = \min\{1, e^{-\gamma d_{ij}}/d_{ij}^{\alpha}\}$$

Where $d_{ij} = |x_i - x_j|, \gamma > 0, \alpha > 2$.

3 Outline of the Achievability Proof

In this section, we state relevant percolation theory results that will be used in constructing a scheme to achieve a strictly positive single-source-single-destination throughput under the interference model. This relies on a percolation theoretic argument of having a large connected cluster of nodes containing the origin. The main steps of the construction are as follows

1. For each $0 < \alpha < 1$, we show the existence of a finite radius $r_\alpha > 0$ such that in the corresponding *Boolean Model* (where two nodes are considered connected iff their euclidean distance is

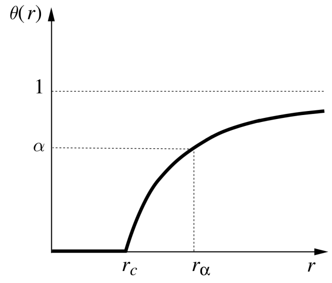

Figure 1: Sketch of Continuous Percolation Function $\theta(r)$

less than or equal to r_α) there exists a *connected cluster*, containing the origin, which includes a given constant α fraction of nodes ($0 < \alpha < 1$) *with high probability*. Since α can be made arbitrarily close to 1, this implies that a random source and destination will lie in that connected cluster with high probability.

2. We then develop a shortest path routing strategy, which relays messages through the connected paths from the source to destination.

3. Since r_α is finite, we can bound total interference power at each slot by scheduling alternative nodes on the shortest path from source to destination. This ensures a constant per-hop throughput and concludes the achievability proof.

In the following, we define and show the existence of a relevant connectivity property stated in Step 1 above. For that we need a couple of classical definitions and theorems on Percolation of Boolean Model of an underlying PPP observed in the whole $\mathbb{R}^2$ plane.

3.1 α-almost Connectivity of $G_n(r)$

Definition 1. *$\theta(r)$ is the probability that origin is contained in an infinite cluster in a boolean model, for an underlying PPP, with connectivity radius r.*

The graph for $\theta(r)$ is sketched as in Fig. 3.1.

Definition 2. *For any $\alpha \in (0,1)$, $G_n(r)$ is said to be α-almost connected if it contains a connected component of at least αn nodes.*

Now we will be relating $\theta(r)$, which is essentially a connectivity property of an infinite graph to α-almost connectivity of a finite n-node graph via the following theorem. This accomplishes step 1 as stated in the outline.

Theorem 2. *Let*

$$r_\alpha = \inf\{r : \theta(r) > \alpha\} \tag{4}$$

Then for any $\alpha \in (0,1)$, if $r > r_\alpha$, then $G_n(r)$ is α-almost connected w.p. 1

Theorem 2 is proved by showing existence of a box $B_{\delta n}$, $(0 < \delta < 1)$ such that it contains atleast αn nodes that are on an infinite cluster in the Boolean Model for an appropriate parameter $r = r_\alpha > 0$ and the box $B_{\delta n}$ is surrounded by a closed circuit as shown in figure 3.1

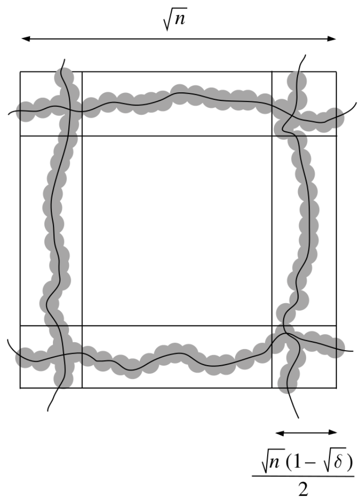

Fig. Existence of the circuit.

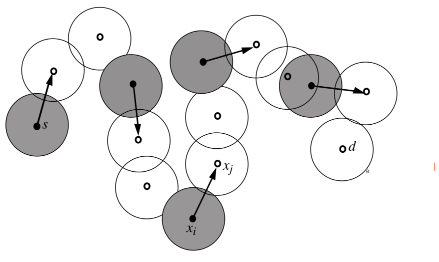

Figure 2: Routing and Scheduling

3.2 Routing and Scheduling Strategy

Once we establish connectivity of αn nodes through Theorem 2, we form a virtual network by connecting two nodes iff they are within an euclidean distance of r_α from each other. We then route packets from source s to destination d via a shortest path in this network using other nodes as relays as shown in Figure 3.2. Because of the shortest-path property, it follows that any ball on the shortest path has overlap with at most two balls, its successor and predecessor, otherwise, we would have taken the short-cut. Thus we use a *periodic schedule of length three* where we transmit from every third node on the shortest path at a time-slot. Note that interfering nodes are atleast $2r$ distance away from any receiver. It can be then easily shown that total interference power is upper bounded by a finite constant and hence per-hop capacity along the shortest path, calculated using Eqn.3 is lower bounded by a non-zero constant, independent of n.

4 Proof of the Converse

Till now we have shown that as $\alpha \to 1$, the rate of communication goes to zero, as $r_\alpha \to \infty$. However, this does not rule out the possibility that a different strategy could achieve a constant rate w.h.p. when $\alpha \to 1$. In the following, we prove a converse result that shows that it is not possible and a non-vanishing rate can not be achieved by all the nodes.

Theorem 3. *For any $R > 0$ and $0 < \alpha < 1$, let $\tilde{A}_n(R, \alpha)$ be the event that there exists a set S_n of at least αn nodes, such that for any $x, y \in S_n$, x can not communicate with y at rate R. Then for any $R > 0$, there exists $\alpha(R) > 0$, such that*

$$\lim_{n\to\infty} \mathbb{P}(\tilde{A}_n(R, \alpha)) = 1$$

We give a different and shorter proof than what is given in the paper [2], under the assumption $\int_0^\infty x l^2(x)dx < \infty$. To this end, we use the following theorem

Theorem 4. ***Campbell's Theorem:*** *Let X be a Poisson process with density λ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying*

$$\int_{\mathbb{R}^2} \min(|f(x)|, 1)dx < \infty \tag{5}$$

Define

$$\Sigma = \sum_{x\in X} f(x) \tag{6}$$

Then we have

$$\mathbb{E}(\Sigma) = \lambda \int_{\mathbb{R}^2} f(x)dx \tag{7}$$

With the help of the previous theorem, we prove the following theorem that will be used in the proof of the converse.

Theorem 5. *If $\int_0^\infty r l^2(r)dr < \infty$ then, for a fixed source s, and for any $\delta > 0, 0 < \epsilon < 1$, there exists a natural number N s.t.*

$$\mathbb{P}\Bigg(\sum_{i:|s-x_i|>N} l^2(s, x_i) < \delta \Bigg) > \epsilon \tag{8}$$

Proof. First we define the sequence of r.v.s $\{Y_i\}_{i=1}^\infty$ as follows

$$Y_n = \sum_{i:|s-x_i|<n} l^2(s, x_i) \tag{9}$$

i.e. Y_n denotes the sum of all gains at the source s due to all nodes that are located within a radius n of the source. Clearly $Y_n \nearrow \Sigma$, where $\Sigma = \sum_{x\in X} l^2(s, x_i)$. Since $\{Y_n\}$s are non-negative and monotone increasing, we can apply *Monotone Convergence Theorem* to conclude that

$$\mathbb{E}(Y_n) \nearrow \mathbb{E}(\Sigma) = 2\pi\lambda \int_{\mathbb{R}^2} r l^2(r)dr = c < \infty \tag{10}$$

Where the last equality follows from Campbell's Theorem and our assumption on the function $l^2()$. Now for any $\delta > 0$,

$$\begin{aligned}
\mathbb{P}(|\Sigma - Y_n| \geq \delta) &\leq \frac{\mathbb{E}(|\Sigma - Y_n|)}{\delta} && (11)\\
&= \frac{\mathbb{E}(\Sigma - Y_n)}{\delta} \quad \text{since, } \Sigma \geq Y_n \text{ w.p.1} && (12)\\
&\searrow 0 \quad \text{as } n \to \infty && (13)
\end{aligned}$$

Where Eqn 11 follows from Markov Inequality and Eqn. 13 follows from Eqn. 10. The above result implies that for any, $\delta > 0, 0 < \epsilon < 1$, we can choose an N s.t.

$$\mathbb{P}\left(\sum_{i:|s-x_i|>N} l^2(s, x_i) < \delta \right) \geq \epsilon \tag{14}$$

This concludes the proof. □

Now, we prove the converse using **Broadcast-Cut** bound from Information Theory, where we will be using Theorem 5 to show that a necessary condition for achievability for any rate $R > 0$ does not hold, thus proving Theorem 3. We now state a special case of Broadcast-Cut bound from Information Theory [1].

Theorem 6. *For all nodes s, d, the achievable rate between them in a multi-terminal AWGN channel is upper bounded as*

$$R(s, d) \leq \log\left(1 + \frac{P \sum_{x \neq s} l^2(s, x)}{N}\right) \tag{15}$$

Proof of Theorem 3

Proof. If a rate $R > 0$ is achievable between source s and destination d, from theorem 6, we necessarily have that

$$\sum_{x \neq s} l^2(s, x) \geq \frac{N}{P}(2^R - 1) = \delta \text{ (say)} \tag{16}$$

Theorem 5 says that, there exists a natural number N s.t. if there is no node within a radius of N from the source, then the above condition is *violated* with positive probability ϵ. But the probability that there is no node within a radius N of the source is **independent** of the above event (since the intersection of the concerned areas for the two events involved is an empty set) and has a positive probability given by $q = e^{-\lambda \pi N^2} \frac{(\lambda \pi N^2)^{n-1}}{(n-1)!} > 0$. Hence for a given source s, there is a strictly positive probability $p = q\epsilon > 0$, such that the necessary condition in 16 is violated for any positive rate $R > 0$. Let Z_n be the number of Poisson Points inside the box B_n for which the above holds. Then by *ergodic theorem*, we have a.s.

$$\lim_{n \to \infty} \frac{Z_n}{n} = p > 0 \tag{17}$$

□

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